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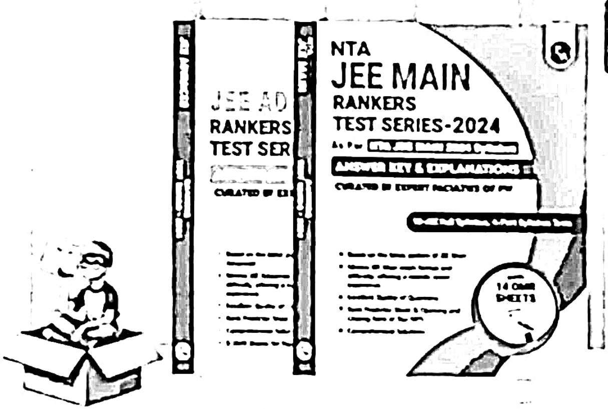
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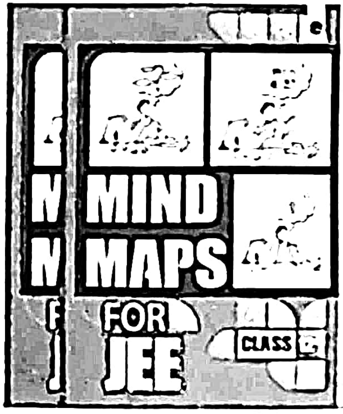
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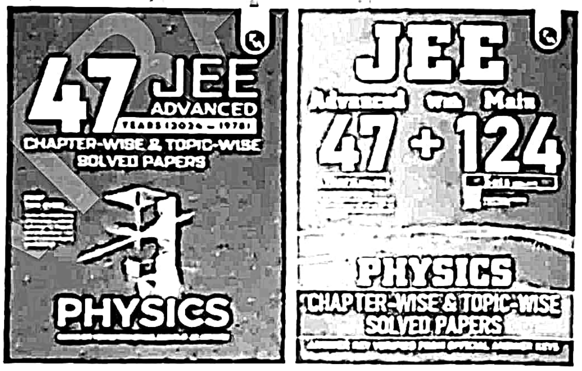
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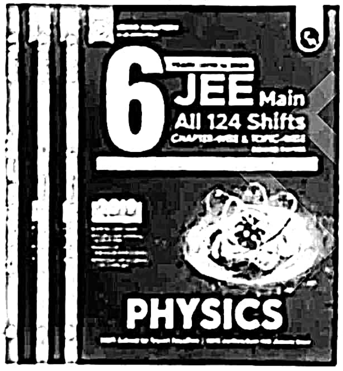
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All 124 Shifts
PHYSICS

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Start and Throughly read your Module

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PHYSICS WALLAH

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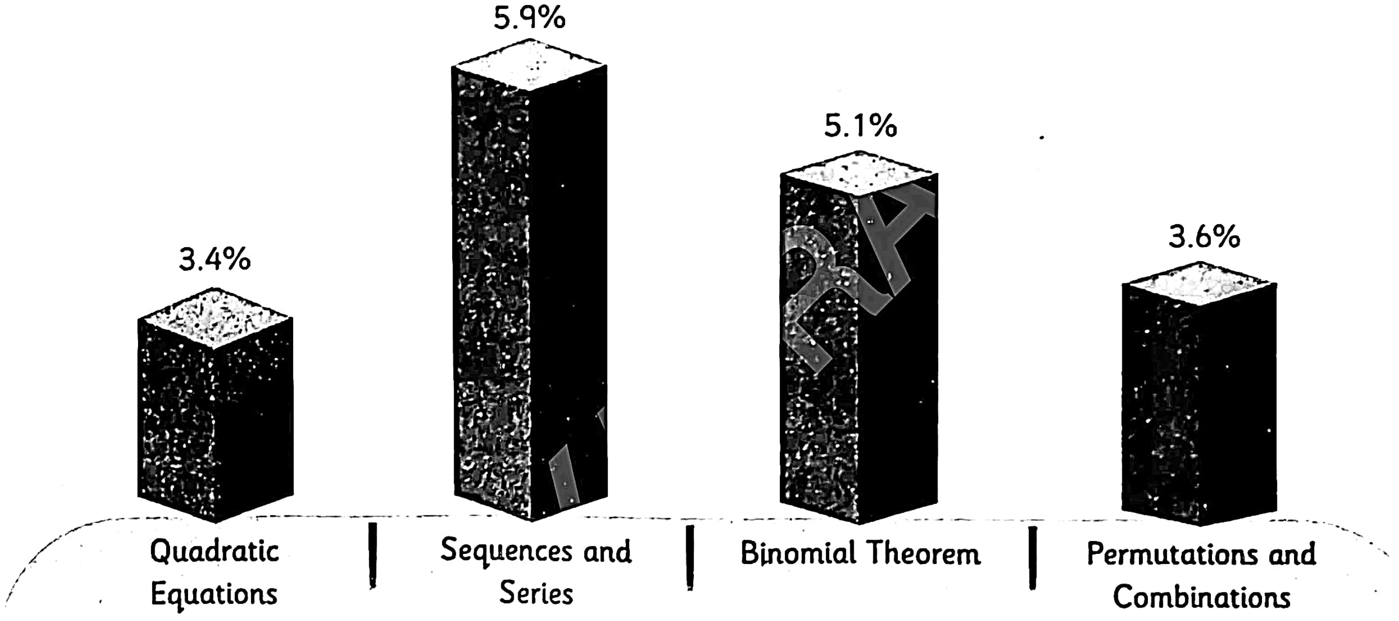

Chapter Wise Weightage (in %) Based on Past 6 Years JEE Main Papers

MATHEMATICS

Module 2

Optimism: In this module for the emotional well being we will practice Optimism along with our regular academic learning.

Credit:--- Divansh.....

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PREFACE

A dedicated team at Physics Wallah (PW) works diligently to provide students with quality content for the **JEE** exam.

From the beginning, the whole content team comprising Associate faculties, DTP operators, Proofreaders and others are involved in shaping the material to their best knowledge and experience to produce powerful content for the students.

Faculties have adopted a new style of presenting the content in easy-to-understand language and have provided the team with their guidance and supervision throughout the creation of this Study Material.

Physics Wallah (PW) strongly believes in conceptual and fun-based learning. PW provides exam-focused content designed to offer clarity and value to students.

A plethora of **JEE Study Material** is available in the market but PW professionals are continuously Working to provide comprehensive study material for our **JEE** students.

This Study Material adopts a multi-faceted approach to master and understanding the concepts by having a rich diversity of questions asked in the examination and equip the students with the knowledge for the competitive exam.

The main objective of the study material is to provide a large number of quality problems with varying cognitive levels to facilitate the teaching-learning of concepts that are presented through the book.

It is widely used by aspirants because of its easy-to-understand language and clear illustrations with diagrams, mnemonics, flow charts and tables.

Students can benefit themselves by attempting the exercise given in this Study Material for the self-assessment and also mastering the basic techniques of problem-solving.

The questions are strictly designed in accordance with the exam relevant topics that helps to develop examination temperament within the students.

By mastering the Physics Wallah (PW) study material curated by the PW team, students can improve their chances of qualifying for the **JEE** exam and achieving a strong rank.

In each chapter, for better understanding, questions have been classified according to the latest syllabus of **JEE**

- The nature and diversity of the questions help students to ace the examination
- Quality questions to strengthen the concept of the topic at the zenith level

BOOK FEATURES

- Detailed theory with Train Your Brain and Concept Application Questions
- "Aarambh" Section for Solved Miscellaneous Examples
- School Level Problems for School Going Students
- "Prarambh" Section for Topicwise, "Prabal" Section for JEE Main Level and "Parikshit" Section for Advance Level MCQs covered in exercises
- Match the Column, Comprehension Based, Numerical Type and Multicorrect questions covered for complete practice
- JEE Past Year Questions
- PW Challengers Problems for Difficult Level Questions
- Elaborated Solutions Booklet you can get separately

Credit:--- Divansh.....

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3P PROGRESSOR

PLANNER

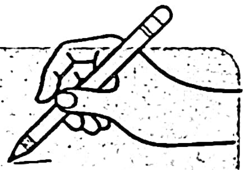
CHAPTER NAMES	Start Date of Module	End Date of Module	Solve More PYQs	Revision Period	Re-revision Period (Before Test Series)
Quadratic Equations					
Sequences and Series					
Binomial Theorem					
Permutations and Combinations					

PROGRESS TRACKING

CHAPTER NAMES	Solved Examples	School Level	Topicwise	Main Level	Advanced Level	PYQs	Challengers
Quadratic Equations	☐	☐	☐	☐	☐	☐	☐
Sequences and Series	☐	☐	☐	☐	☐	☐	☐
Binomial Theorem	☐	☐	☐	☐	☐	☐	☐
Permutations and Combinations	☐	☐	☐	☐	☐	☐	☐

PREP POINTERS

Note down the topic you want to revise or section you want to re attempt while your preparation.



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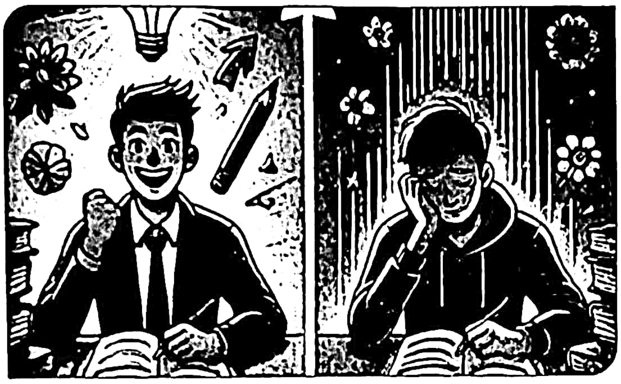
The Power of Optimism: Turning Challenges into Opportunities

Optimism is the fuel for perseverance. Let hope and positivity guide you through challenges, turning every obstacle into an opportunity for success. For a JEE aspirant, it means believing in your ability to overcome challenges, staying positive about progress, and trusting that every effort will bring you closer to your goal. It fuels persistence and resilience during the journey.

> "Optimism is the faith that leads to achievement. Nothing can be done without hope and confidence."
> – Helen Keller



Without Optimism

- Without optimism, setbacks can feel overwhelming, leading to self-doubt and difficulty in recovering from failures.
- A pessimistic mindset reduces focus and productivity, often leaving concepts incomplete or misunderstood.

With Optimism

- Optimism transforms challenges into growth opportunities, fostering hope and resilience during setbacks. It enhances focus and energy, even during long hours of study.
- Optimistic students tackle problems with a solution-oriented mindset, learning from mistakes and consistently making progress.

SCENARIOS

Without Optimism		With Optimism
"I failed again. I'll never be able to do this."	After a Difficult Mock Test	"This was tough, but it's an opportunity to learn. I'll improve next time."
"I'm not good enough. Maybe I should give up."	Feedback from a Teacher	"This feedback will help me grow. I know where I need to improve."
"I'm exhausted. I can't keep this up."	Studying Long Hours	"The hard work will pay off. Every hour counts towards my goal."
"This is too hard. I don't understand anything."	Facing a Tough Concept	"This is challenging, but I will break it down step by step."
"I'm way behind. I'll never catch up."	Comparing Scores with Friends	"Everyone has their own pace. I'll focus on my progress."
"I'm missing out on fun. This is too hard."	Missing a Social Event for Study	"I'm investing in my future. The sacrifices now will pay off later."
"Why is everything going wrong? I can't deal with this."	Unexpected Challenges During Preparation	"Challenges are a part of the journey. I'll adapt and find a way forward."
"I'm stuck in this routine. It's getting monotonous."	Daily Study Routine	"Sticking to my routine is helping me build discipline and getting me closer to my goal."

This demonstrates how adopting a mindset of **Optimism** can help students navigate the stresses and challenges of exam preparation with more resilience and positivity. It highlights how a shift in perspective can transform obstacles into opportunities for growth and improvement.

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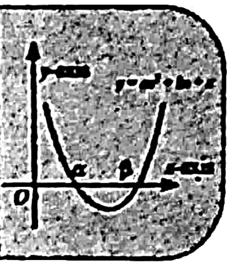
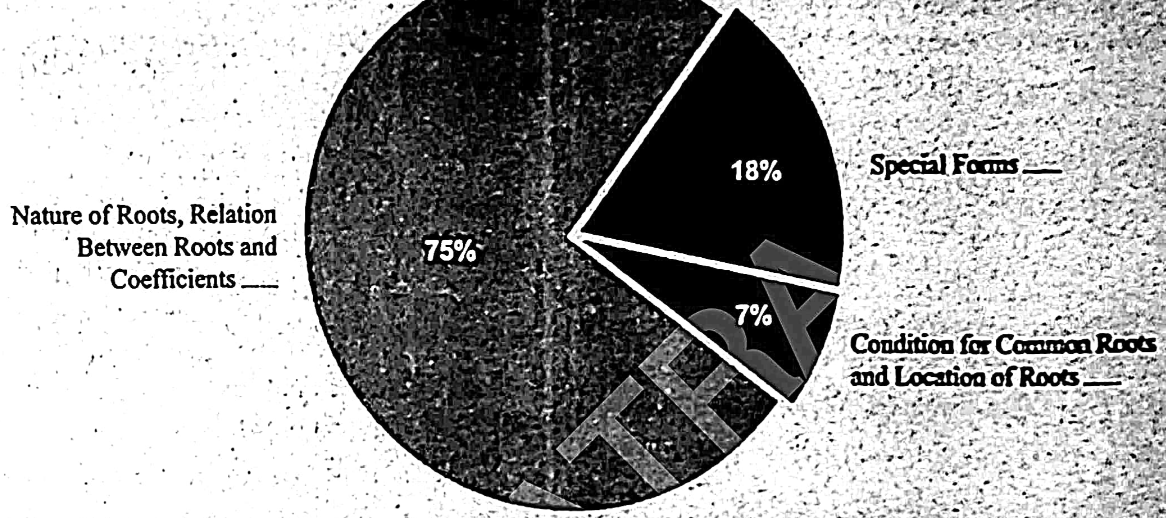
CHAPTER

5

Quadratic Equations



Topicwise Weightage of JEE Main 6 Years Paper (124 Sets)



"How's the Josh?" for these Topics: *Mark your confidence level in the blank space around the topic (Low-L, Medium-M, High-H).*

INTRODUCTION

General Real Polynomial

An expression written as, $a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$, where $a_0, a_1, a_2, ..., a_n \in R$ is called a real polynomial of degree n where $(a_n \neq 0, n \in W)$.

$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 = 0$ is known as **polynomial equation.**

Quadratic Polynomial with Real Coefficients

Polynomial of form $ax^2 + bx + c$ where $a \neq 0$ and $a, b, c \in R$ is a quadratic polynomial with real coefficients.

Note: If $a = 0$, then above expression reduces to $bx + c$ which is a linear polynomial provided $b \neq 0$.

QUADRATIC EQUATIONS WITH REAL COEFFICIENTS

Second-degree equation of form $ax^2 + bx + c = 0$, where $a \neq 0$ and $a, b, c \in R$ is quadratic equation with real coefficients.

1. The values of x which satisfy the quadratic equation are called roots of the quadratic equation.
2. Generally, three methods factorization method, completing the square method and quadratic formula are use to solve the quadratic equation.
3. This equation has two roots which are given by $x = \frac{-b \pm \sqrt{D}}{2a}$, where D (or Δ) $= b^2 - 4ac$ is called as discriminant of the quadratic equation.

EQUATION V/S IDENTITY

A quadratic equation is satisfied by exactly two values of 'x' which may be real or imaginary. The equation, $ax^2 + bx + c = 0$ is a quadratic equation if $a \neq 0$ and An identity if $a = b = c = 0$ (Infinite Roots).

$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-a)(b-c)} = 1$ has roots as a, b, c so it is identity because it can not have more than 2 roots due to quadratic nature.

Credit:--- Divansh.....

Train Your Brain

Example 1: The roots of the quadratic equation $3x^2 - 5x - 12 = 0$ are

(*a*) $-\frac{4}{3}, 3$ (*b*) $-\frac{3}{4}, 3$

(*c*) $-\frac{5}{4}, 2$ (*d*) $-\frac{4}{5}, 2$

Sol. (*a*) $x = \frac{5 \pm \sqrt{25 - 4(3)(-12)}}{2 \times 3}$ i.e. $x = -\frac{4}{3}, 3$

Alternate method (using factorization method)

$3x^2 - 9x + 4x - 12 = 0$

$3x(x - 3) + 4(x - 3) = 0$

$(3x + 4)(x - 3) = 0,\ x = -\frac{4}{3}, 3$

Example 2: The quadratic equation whose roots are $1+\sqrt{2}, 1-\sqrt{2}$ is

(*a*) $x^2 + 2x - 1 = 0$ (*b*) $x^2 - 2x - 1 = 0$

(*c*) $x^2 - 2x + 1 = 0$ (*d*) $x^2 + 2x + 1 = 0$

Sol. (*b*) $(x-(1+\sqrt{2}))(x-(1-\sqrt{2})) = 0$

i.e. $x^2 - 2x + (-1) = 0$

$x^2 - 2x - 1 = 0$ (or) $x = 1 + \sqrt{2}$

$x = 1 = \sqrt{2}(x-1)^2 = 2$

$x^2 - 2x - 1 = 0$

Example 3: If a, b, c are positive real numbers then the number of real roots of the equation $ax^2 + b|x| + c = 0$ is

(*a*) 2 (*b*) 4 (*c*) 0 (*d*) –1

Sol. (*c*) $a|x|^2 + b|x| + c = 0$

using quadratic formula,

$$|x| = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now $|x| \neq \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0$ (since, a,b,c are +ve)

also, since a, b, c are positive real number

$\therefore b^2 - 4ac < b^2$

$\Rightarrow \sqrt{b^2 - 4ac} - b < 0 \Rightarrow |x| \neq \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

$\therefore$ No. of real roots = 0

Example 4: Find the value of m such that $(m^2 - 3m + 2)x^2 + (m^2 - 5m + 4)x + (m^2 - 7m + 6) = 0$ is an identify.

(*a*) 1 (*b*) 2 (*c*) –2 (*d*) –1

Sol. (*a*) $m^2 - 3m + 2 = 0$

m = 1, 2

$m^2 - 5m + 4 = 0 \Rightarrow m = 1, 4$

$m^2 - 7m + 6 = 0 \Rightarrow m = 1, 6$

acceptable value of $m = 1$

Concept Application

1. If $3^{1+x} + 3^{1-x} = 10$ then the values of x are

   (*a*) 1, –1 (*b*) 1, 0

   (*c*) 1, 2 (*d*) –1, –2

2. If the roots of the quadratic equation $2024x^2 + 2025x + 1 = 0$ are α and β, then $2024\,\alpha\beta =$

   (*a*) 1 (*b*) –1 (*c*) 2024 (*d*) 2025

3. Number of real roots of the equation $3x^2 + 4|x| + 5 = 0$ are

   (*a*) 2 (*b*) 4 (*c*) 0 (*d*) –1

RELATION BETWEEN ROOTS AND CO-EFFICIENTS

If α, β are the roots of quadratic equation,

$ax^2 + bx + c = 0$ ... (*i*)

sum of the roots $= \alpha + \beta = -\frac{b}{a} = -\frac{\text{coefficent of } x}{\text{coefficient of } x^2}$

product of the roots $= \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

$\therefore$ Equation (*i*) can be written as

$a(x^2 - (\alpha + \beta)x + \alpha\beta) = 0$

$\Rightarrow$ $a(x^2 -$ (sum of the roots) x + (product of the roots)) = 0

SYMMETRIC FUNCTION OF THE ROOTS

A function of α and β is said to be a symmetric function if it remains unchanged when α and β are interchanged..

(*i*) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(*ii*) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)$

(*iii*) $\alpha^4 + \beta^4 = (\alpha^3 + \beta^3)\,(\alpha + \beta) - \alpha\beta\,(\alpha^2 + \beta^2)$

(*iv*) $\alpha^5 + \beta^5 = (\alpha^3 + \beta^3)\,(\alpha^2 + \beta^2) - \alpha^2\beta^2(\alpha + \beta)$

(*v*) $|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

Train Your Brain

Example 5: If α and β are the roots of $ax^2 + bx + c = 0$ and γ, δ are those of $lx^2 + mx + n = 0$, then the sum of coefficient of x^2 and x in the quadratic equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ is (if leading coefficient is unity.)

(*a*) $al(al + bm)$ (*b*) $-al(al - bm)$

(*c*) $al(al - bm)$ (*d*) $-al(al + bm)$

Sol. (*c*) $ax^2 + bx + c = 0$ has roots α and β

$\therefore \alpha + \beta = -\frac{b}{a};\quad \alpha\beta = \frac{c}{a}$

Credit:--- Divansh.....

$lx^2 + mx + n = 0$ has roots γ and δ

$$\therefore\ \gamma+\delta=-\frac{m}{l};\quad \gamma\delta=\frac{n}{l}$$

The required quadratic equation is

$x^2 - (\alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma)x + (\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma) = 0$

where $\alpha\gamma+\beta\delta+\alpha\delta+\beta\gamma=(\alpha+\beta)(\gamma+\delta)=\frac{b}{a}.\frac{m}{l}$

$$(\alpha\gamma+\beta\delta)(\alpha\delta+\beta\gamma)=\frac{n}{l}\left(\frac{b^2-2ac}{a^2}\right)+\frac{c}{a}\left(\frac{m^2-2nl}{l^2}\right)$$

$$=\frac{ln\,(b^2-2ac)+ac(m^2-2nl)}{a^2l^2}$$

$$\therefore\ a^2l^2\cdot x^2-ablm\,x+ln\,(b^2-2ac)+ac(m^2-2nl)=0$$

$\therefore$ sum of coefficient of x^2 and $x = al(al - bm)$.

Example 6: If α, β are the roots of $ax^2 + bx + c = 0$ and $c \neq 0$ then the value of $\frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2}$ in terms of a, b, c is

(*a*) $\frac{b^2-2ac}{ac}$ (*b*) $\frac{2ac-b^2}{ac}$

(*c*) $\frac{b^2-2ac}{a^2c^2}$ (*d*) $\frac{b^2}{a^2c}$

Sol. (*c*) $\because$ α is a root of $ax^2 + bx + c = 0$

$$\Rightarrow a\alpha^2+b\alpha=-c \Rightarrow a\alpha+b=\frac{-c}{\alpha},\ \text{similarly } a\beta+b=\frac{-c}{\beta}$$

$$\therefore\ \frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{c^2}=\frac{b^2-2ac}{a^2c^2}.$$

Example 7: If α, β are the roots of $ax^2 + bx + c = 0$, then the equation whose roots are $2 + \alpha$ and $2 + \beta$ is

(*a*) $ax^2 + (b - 4a)x + (4a - 2b + c) = 0$

(*b*) $ax^2 - (b - 4a)x + (4a - 2b + c) = 0$

(*c*) $ax^2 + (b - 4a)x - (4a - 2b + c) = 0$

(*d*) $ax^2 - (b - 4a)x - (4a - 2b + c) = 0$

Sol. (*a*) $f(x - 2) = 0 \Rightarrow a(x - 2)^2 + b(x - 2) + c = 0$

$\Rightarrow ax^2 + x(b - 4a) + 4a - 2b + c = 0$

Alternate method

α, β are roots of

$ax^2 + bc + c = 0$

$$\therefore \alpha+\beta=\frac{-b}{a},\quad \alpha\beta=\frac{c}{a}$$

Equation with roots $(2 + \alpha)$ and $(2 + \beta)$ will be

$x^2 - (2 + \beta + 2 + \alpha)x + (2 + \alpha)(2 + \beta) = 0$

$x^2 - (4 + \alpha + \beta)x + [4 + 2(\alpha + \beta) + \alpha\beta] = 0$

$$x^2-\left(4-\frac{b}{a}\right)x+\left[4-\frac{2b}{a}+\frac{c}{a}\right]=0$$

$ax^2 + (b - 4a)x + (4a - 2b + c) = 0$

Example 8: If each root of the equation $x^2 + 11x + 13 = 0$ is diminished by 4, then sum of the roots of the transfered equation is

(*a*) 19 (*b*) −19 (*c*) 17 (*d*) −17

Sol. (*b*) $f(x + 4) = 0 \Rightarrow (x + 4)^2 + 11(x + 4) + 13 = 0$

$\Rightarrow x^2 + 19x + 73 = 0$

$$\therefore\ \text{sum of the roots}=-\frac{b}{a}=-\frac{\text{coefficient of } x}{\text{coefficient of } x^2}=-19$$

Example 9: If α, β are the roots of $4x^2 + 7x + 2 = 0$, then the equation whose roots are α^2, β^2 is

(*a*) $16x^2 + 33x + 4 = 0$ (*b*) $16x^2 - 33x - 4 = 0$

(*c*) $16x^2 + 33x - 4 = 0$ (*d*) $16x^2 - 33x + 4 = 0$

Sol. (*d*) $f(x) = 4x^2 + 7x + 2 = 0$

The required equation is $f\left(\sqrt{x}\right)=0.$

$$\Rightarrow\ 4\left(\sqrt{x}\right)^2+7\sqrt{x}+2=0\Rightarrow 4x+2=-7\sqrt{x}$$

$$\Rightarrow\ (4x+2)^2=\left(-7\sqrt{x}\right)^2\Rightarrow 16x^2-33x+4=0$$

Concept Application

4. If α, β are roots of the equation $2x^2 + 6x + b = 0\,(b < 0)$ then $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ is less than

   (*a*) 2 (*b*) −2 (*c*) 18 (*d*) 0

5. If α, β are roots of $x^2 + p(x + 1) + C = 0$ then $(\alpha + 1)(\beta + 1) =$

   (*a*) $1 - C$ (*b*) $1 + C$ (*c*) $C - 1$ (*d*) C

6. If the roots of quadratic equation $ax^2 + bx + c = 0$ are α and β. Then $\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2=$

   (*a*) $\frac{b^2}{a^2}$ (*b*) $\frac{b^2}{c^2}$ (*c*) $\frac{a^2}{b^2}$ (*d*) $\frac{c^2}{b^2}$

7. Let two numbers have arithmetic means 9 and geometric mean 4. Then these numbers are the roots of the quadratic equation

   (*a*) $x^2 + 18x + 16 = 0$ (*b*) $x^2 - 18x - 16 = 0$

   (*c*) $x^2 + 18x - 16 = 0$ (*d*) $x^2 - 18x + 16 = 0$

8. If α, β are root of $ax^2 + bx + c = 0$ then $\frac{1}{\alpha^3}+\frac{1}{\beta^3}=$

   (*a*) $\frac{3abc-b^3}{a^3}$ (*b*) $\frac{3abc-b^3}{a^3c}$

   (*c*) $\frac{3abc-b^3}{c^3}$ (*d*) $\frac{b^3-2ac}{ac}$

9. If α, β are the roots of the equation $x^2 + 3x + 2 = 0$. Then $\alpha^5 + \beta^5 =$

   (*a*) 33 (*b*) −32 (*c*) 35 (*d*) −33

Credit:--- Divansh.....

NATURE OF ROOTS

- ❖ Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then;
  - (*i*) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal)
  - (*ii*) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)
  - (*iii*) $D < 0 \Leftrightarrow$ roots are imaginary
  - ✦ If $p + iq$ is one root of a quadratic equation, then the other must be it's conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$)
- ❖ Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then ;
  - (*i*) If $D > 0$ & is a perfect square, then roots are rational & unequal.
  - (*ii*) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then the other root must be the conjugate of it i.e. $\beta = p - \sqrt{q}$ & vice versa.

If $\Delta = b^2 - 4ac > 0$.

Now consider the following cases.

Case		Nature of roots
Case-I	$a > 0, b > 0, c > 0$ $\Rightarrow \alpha + \beta < 0,$ $\alpha\beta > 0$	Both roots are negative.
Case-II	$a > 0, b > 0, c < 0$ $\Rightarrow \alpha + \beta < 0,$ $\alpha\beta < 0$	Both roots are opposite in sign; Magnitude of negative root is more than the magnitude of positive root.
Case-III	$a > 0, b < 0, c > 0$ $\Rightarrow \alpha + \beta > 0,$ $\alpha\beta > 0$	Both roots are positive.
Case-IV	$a > 0, b < 0, c < 0$ $\Rightarrow \alpha + \beta > 0,$ $\alpha\beta < 0$	Roots are opposite in sign. Magnitude of positive root is more than magnitude of negative root.

PTR (Points to Remember)

(*i*) Roots are rational $\Leftrightarrow$ D is a perfect square.

(*ii*) Roots are irrational $\Leftrightarrow$ D is positive but not a perfect square.

(*iii*) If $a + b + c = 0$, then 1 is a root of the equation $ax^2 + bx + c = 0$.

(*iv*) If a and c are of opposite sign, the roots must be of opposite sign.

(*v*) If the roots are equal in magnitude but opposite in sign, then $b = 0, ac < 0$.

(*vi*) If the roots are reciprocal of each other, then $c = a$.

(*vii*) If $a = 1$, $b, c \in Z$ and the roots are rational numbers, then these roots must be integers.

(*viii*) If $P(x) = a_1x^2 + b_1x + c_1$ and $Q(x) = a_2x^2 + b_2x + c_2$, then $P(x).\,Q(x) = 0$ have atleast two real roots if $D_1 + D_2 > 0$.

TRANSFORMATION OF EQUATION

If $f(x) = 0$ & $g(x) = 0$ are two polynomial equation having some common root(s) then those common root(s) is/are also the root(s) of $h(x) \equiv a f(x) + bg(x) = 0$.

Train Your Brain

Example 10: If the sum of all the values of m for which the quadratic equation $5x^2 - 4x + 2 + m(4x^2 - 2x - 1) = 0$ have equal roots is l, then $-5l =$

(*a*) 1 (*b*) – 1 (*c*) 5 (*d*) –5

Sol. (*a*) $5x^2 - 4x + 2 + m(4x^2 - 2x - 1) = 0$

$$(5 + 4m)x^2 - 2(m + 2)x + (2 - m) = 0 \quad \text{... (i)}$$

For equal roots, the discriminant of (i) should be equal to zero.

$$\therefore\ 4(m + 2)^2 - 4(5 + 4m)(2 - m) = 0$$

$$\Rightarrow m^2 + 4m + 4 - (10 + 8m - 5m - 4m^2) = 0$$

$$\Rightarrow 5m^2 + m - 6 = 0 \Rightarrow (5m + 6)(m - 1) = 0$$

$$\Rightarrow m = -\frac{6}{5} \quad \text{or} \quad m = 1$$

Example 11: The roots of the quadratic equation $x^2 + x - 1 = 0$ are

(*a*) Rational

(*b*) Irrational and conjugate

(*c*) Real

(*d*) Complex

Sol. (*b*) $\Delta = (1)^2 - 4(1)(-1) = 5$ is not a perfect square.

$\Rightarrow$ roots are irrational and more over conjugate

Example 12: The roots of the quadratic equation $2x^2 + x + 6 = 0$ are

(*a*) Rational

(*b*) Real number

(*c*) Irrational and conjugate complex number

(*d*) Complex

Sol. (*c*) $\Delta = 1 - 4(2)(6) = -47 < 0$ is not a perfect square.

$\Rightarrow$ roots are irrational and more over conjugate complex number

Example 13: If the roots of the equation $x^2 - 15 - m(2x - 8) = 0$ are equal then sum of all the values of m is

(*a*) 8 (*b*) 15

(*c*) 3 (*d*) 5

Sol. (*a*) $x^2 - 15 - m(2x - 8) = 0$

$x^2 - 2mx + 8m - 15 = 0$

$\Delta = 0 \Rightarrow (-2m)^2 - 4 \times 1 \times (8m - 15) = 0$

$m^2 - 8m + 15 = 0$

$(m - 3)(m - 5) = 0, m = 3; m = 5$

Example 14: If a, b, c are real, the roots of $a^2x^2 + b^2x^2 - 2abx - 2bcx + b^2 + c^2 = 0$ are real and equal then a, b, c are in

(*a*) A.P (*b*) G.P

(*c*) H.P (*d*) A.G.P

Sol. (*b*) $a^2x^2 + b^2x^2 - 2abx - 2bcx + b^2 + c^2 = 0$

$a^2x^2 - 2abx + b^2 + b^2x^2 - 2bcx + c^2 = 0$

$(ax - b)^2 + (bx - c)^2 = 0$

$ax - b = 0;\ bx - c = 0$

$x = \frac{b}{a};\ x = \frac{c}{b}$

$\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$

$\therefore$ a, b, c are in G.P.

Example 15: If the roots of equation $(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0$ are equal then

(*a*) $b = 0$ (or) $a^3 + b^3 + c^3 - 3abc = 0$

(*b*) $b = 1$ (or) $a^3 + b^3 + c^3 - 3abc = 0$

(*c*) $b = 0$ (or) $a^3 + b^3 + c^3 + 3abc = 0$

(*d*) $b = 1$ (or) $a^3 + b^3 + c^3 + 3abc = 0$

Sol. (*a*) $(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0$

roots are equal $\Rightarrow \Delta = 0$

$\Rightarrow 4(b^2 - ca)^2 - 4(a^2 - bc)(c^2 - ab) = 0$

$\Rightarrow b^4 - 2ab^2c + a^3b + bc^3 - ab^2c) = 0$

$\Rightarrow b = 0$ (or) $a^3 + b^3 + c^3 - 3abc = 0$

Example 16: A quadratic equation $p(x) = 0$ has $1 + \sqrt{5}$ and $1 - \sqrt{5}$ as roots and it satisfies $p(1) = 2$, then the quadratic polynomial is.

(*a*) $2/5\,(x^2 - 2x - 4)$ (*b*) $-2/5\,(x^2 - 2x - 4)$

(*c*) $-2/5\,(x^2 + 2x - 4)$ (*d*) $2/5\,(x^2 + 2x - 4)$

Sol. (*b*) Sum of the roots = 2, product of the roots = – 4

$\therefore$ let $p(x) = a(x^2 - 2x - 4) \Rightarrow p(1) = 2$

$\Rightarrow 2 = a(1^2 - 2 \cdot 1 - 4) \Rightarrow a = -2/5$

$\therefore$ $p(x) = -2/5\,(x^2 - 2x - 4)$

Example 17: The quadratic equation $x^2 + mx + n = 0$ has roots which are as twice as those of $x^2 + px + m = 0$, m, n, p are non zero, then the value of $\frac{n}{p} =$

(*a*) 8 (*b*) 5 (*c*) 3 (*d*) 4

Sol. (*a*) $x^2 + mx + n = 0 \begin{array}{l} \nearrow 2\alpha \\ \searrow 2\beta \end{array}$

and $x^2 + px + m = 0 \begin{array}{l} \nearrow \alpha \\ \searrow \beta \end{array}$

$2(\alpha + \beta) = -m$ ... (*i*)

$4\,\alpha\beta = n$ ... (*ii*)

and $\alpha + \beta = -p$ ... (*iii*)

$\alpha\beta = m$ ... (*iv*)

$\therefore$ from (*i*) and (*iii*)

$\Rightarrow 2p = m$

and from (ii) and (iv)

$\Rightarrow 4m = n \Rightarrow \frac{n}{p} = \frac{4m}{m/2} = 8$

Example 18: The value of m for which the quadratic equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots is

(*a*) 0, 3 (*b*) 0, –3 (*c*) 9, 0 (*d*) –9, 0

Sol. (*a*) Given equation is

$(1 + m)\,x^2 - 2(1 + 3m)x + (1 + 8m) = 0$

Roots of equation will be equal if, $D = 0$.

$\Rightarrow 4(1 + 3m)^2 - 4(1 + m)\,(1 + 8m) = 0$

$\Rightarrow 4(1 + 9m^2 + 6m - 1 - 9m - 8m^2) = 0$

$\Rightarrow m^2 - 3m = 0$ or, $m(m - 3) = 0$

$\therefore$ $m = 0, 3$.

Example 19: Sum of all the integral values of a for which the quadratic equation $(x - a)\,(x - 10) + 1 = 0$ has integral roots is

(*a*) 22 (*b*) 20 (*c*) 12 (*d*) 8

Sol. (*b*) Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$.

For integral roots D should be a perfect square.

$D = a^2 - 20a + 96$.

$\Rightarrow D = (a - 10)^2 - 4 \Rightarrow 4 = (a - 10)^2 - D$

If D is a perfect square means difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow (a - 10) = \pm 2; \Rightarrow a = 12, 8$

Concept Application

10. If the expression $x^2 - (5m - 2)x + (4m^2 + 10m + 25)$ can be expressed as a perfect square, then $m =$

(*a*) $\frac{8}{3}$ or 4 (*b*) $-\frac{8}{3}$ or 4 (*c*) $\frac{4}{3}$ or 8 (*d*) $-\frac{4}{3}$ or 8

11. If the roots of the quadratic equation $x^2 - 4x - \log_3 a = 0$ are real, then the least value of a is

(*a*) 81 (*b*) $\frac{1}{81}$ (*c*) $\frac{1}{64}$ (*d*) 9

12. If the roots of the equation $(a^2 + b^2)x^2 + 2(bc + ad)x + (c^2 + d^2) = 0$ are real then a^2, bd, c^2 are in

(*a*) A.P. (*b*) G.P. (*c*) H.P (*d*) A.G.P.

13. The roots of the equation $x^2 - 2\sqrt{3}x + 3 = 0$ are

(*a*) rational and equal (*b*) rational and not equal

(*c*) irrational (*d*) imaginary

14. If one root of the quadratic equation $ax^2 + 3x + 5 = 0$ is 1, then the second root is

(*a*) $\frac{5}{3}$ (*b*) $\frac{-5}{3}$ (*c*) $\frac{5}{16}$ (*d*) $\frac{-5}{8}$

COMMON ROOTS

One Root Common

If $\alpha \neq 0$ is a common root of the equation

$a_1x^2 + b_1x + c_1 = 0$ ... (*i*)

and $a_2x^2 + b_2x + c_2 = 0$ ... (*ii*)

Then we have

$a_1\alpha^2 + b_1\alpha + c_1 = 0$ and $a_2\alpha^2 + b_2\alpha + c_2 = 0$

These give $\dfrac{\alpha^2}{b_1c_2 - b_2c_1} = \dfrac{\alpha}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1} (a_1b_2 - a_2b_1 \neq 0)$.

Thus, the required condition for one common root is $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$ and the value of the common root is

$\alpha = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$ *or* $\dfrac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$.

Both Roots Common

If the equations (i) and (ii) have both roots common, then these equations will be identical. Thus the required condition for both roots common is

$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ (If no root is equal to zero)

PTR (Points to Remember)

(*i*) To find the common root of two equations, make the coefficient of second degree terms in two equations equal and subtract. The value of x so obtained is the required common root.

(*ii*) If two quadratic equations with real and rational coefficients have a common imaginary or irrational roots, then both roots will be common and the two equations will be identical. The required condition is

$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

(*iii*) If α is a repeated root of the quadratic equation $f(x) = ax^2 + bx + c = 0$,
Then α is also a root of the equation $f'(x) = 0$.

Train Your Brain

Example 20: The value of m and n in order that the equations $mx^2 + 5x + 2 = 0$ and $3x^2 + 10x + n = 0$ may have both the roots common is

(*a*) $\dfrac{4}{3}, 2$ (*b*) $\dfrac{3}{2}, 4$

(*c*) $\dfrac{3}{4}, 2$ (*d*) None of these

Sol. (*b*) The equations are $mx^2 + 5x + 2 = 0$ and $3x^2 + 10x + n = 0$.
Since they have both the roots common,

$\dfrac{m}{3} = \dfrac{5}{10} = \dfrac{2}{n} \because \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_1}$

From the first-relation, $m = \dfrac{15}{10} = \dfrac{3}{2}$.

From the last-relation, $n = 4$.

Example 21: If the quadratic equations, $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then

(*a*) $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$

(*b*) $b - c + 1 = 0$ or $b^2 - c^2 + 1 = bc + b + c$

(*c*) $b + c + 1 = 0$ or $b^2 - c^2 + 1 = bc - b + c$

(*d*) None of these

Sol. (*a*) $\alpha^2 + b\alpha + c = 0$ ... (*i*)

$b\alpha^2 + c\alpha + 1 = 0$ ... (*ii*)

$\therefore \dfrac{\alpha^2}{b - c^2} = \dfrac{\alpha}{bc - 1} = \dfrac{1}{c - b^2}$

$\Rightarrow \alpha^2 = \dfrac{b - c^2}{c - b^2}$ or $\alpha = \dfrac{bc - 1}{c - b^2}$

$\Rightarrow (bc - 1)^2 = (b - c^2)(c - b^2)$

or $b^3 + c^3 + 1 - 3bc = 0$

$\Rightarrow (b + c + 1)(b^2 + c^2 + 1 - bc - c - b) = 0$

$\Rightarrow b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$

Example 22: If $x^2 + 4ax + 3 = 0$ and $2x^2 + 3ax - 9 = 0$ have a common root then the value of $|a|$ is

(*a*) 1 (*b*) 0

(*c*) –1 (*d*) None of these

Sol. (*a*) $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$

$(6 + 9)^2 = (3a - 8a)(-36a) \Rightarrow a = \pm 1$

Example 23: If $x^2 + bx + c = 0, x^2 + cx + b = 0 \ (b \neq c)$, have a common root then

(*a*) $b - c + 1 = 0$ (*b*) $-b + c - 1 = 0$

(*c*) $b + c + 1 = 0$ (*d*) $b - c - 1 = 0$

Sol. (*c*) Let α be the common root

$\Rightarrow \alpha^2 + b\alpha + c = 0$ ... (*i*)

$\alpha^2 + c\alpha + b = 0$ ... (*ii*)

(i) – (ii) $\Rightarrow \alpha = 1$

Substitute in (1)

$\Rightarrow b + c + 1 = 0$

Example 24: If the equation $k(6x^2 + 3) + rx + (2x^2 - 1) = 0$ and $6k(2x^2 + 1) + px + (4x^2 - 2) = 0$ have both roots common, then the value of p/r is

(*a*) $\dfrac{1}{3}$ (*b*) 2 (*c*) $\dfrac{1}{2}$ (*d*) $\dfrac{1}{-3}$

Sol. (*b*) $(6k + 2)x^2 + rs + 3k - 1 = 0$ ... (*i*)

$(12k + 4)x^2 + px + 6k - 2 = 0$ ... (*ii*)

Since the roots of (*i*) and (*ii*) and equal,

$\dfrac{6k + 2}{12k + 4} = \dfrac{r}{p} = \dfrac{3k - 1}{6k - 2}$

$\therefore \dfrac{r}{p} = \dfrac{1}{2}$

Concept Application

15. If the equation $x^2 + ax + b = 0$ and $x^2 + bx + a = 0 (a \neq b)$ have a common root then $a + b =$

(*a*) -1 (*b*) 0 (*c*) 1 (*d*) 2

16. If $(x - 2)$ is a common factor of the expressions $x^2 + ax + b$ and $x^2 + cx + d$, then $\frac{b-d}{c-a}$

(*a*) -2 (*b*) -1 (*c*) 1 (*d*) 2

THEORY OF EQUATIONS

If $\alpha_1, \alpha_2, \alpha_3, \ldots\alpha_n$ are the roots of the equation; $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$ where $a_0, a_1, \ldots a_n$ are all real & $a_0 \neq 0$ then,

$$\Sigma\,\alpha_1 = -\frac{a_1}{a_0},\ \Sigma\,\alpha_1\alpha_2 = +\frac{a_2}{a_0},\ \Sigma\,\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, \ldots.., \alpha_1\alpha_2\alpha_3 \ldots\ldots..\alpha_n = (-1)^n\frac{a_n}{a_0}$$

GRAPHS & SIGN OF REAL QUADRATIC EXPRESSIONS

$$\text{Let } y = ax^2 + bx + c = a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right) \quad \ldots \text{(i)}$$

1. Represents a parabola with vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ and axis of the parabola is $x = \frac{-b}{2a}$

If $a > 0$, the parabola opens upward while if $a < 0$, the parabola opens downward. The parabola cuts the x-axis at points corresponding to roots of $ax^2 + bx + c = 0$. If this equation has

(*i*) $D > 0$, the parabola cuts x-axis at two real and distinct points.

(*ii*) $D = 0$, the parabola touches x-axis at $x = \frac{-b}{2a}$.

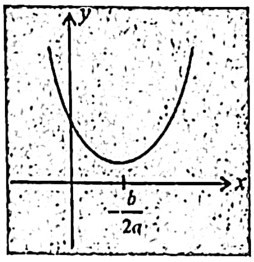
(*iii*) $D < 0$, then;

If $a > 0$, parabola lies above x-axis.

If $a < 0$, parabola lies below x-axis.

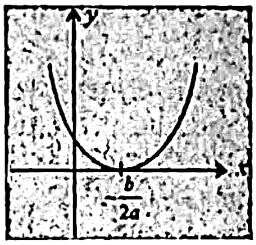
Let $f(x) = ax^2 + bx + c$ and $\alpha, \beta : \alpha < \beta$, be its roots

(*i*) $a > 0$ and $D < 0 \Leftrightarrow f(x) > 0\ \forall\, x \in R$.



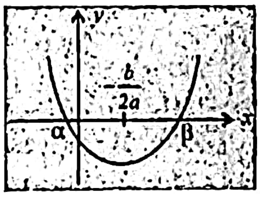
α and β are complex conjugates

(*ii*) $a > 0$ and $D = 0 \Leftrightarrow f(x) \geq 0\ \forall\, x \in R$



$f(x) = 0$ at $x = \frac{-b}{2a}$

(*iii*) $a > 0$ and $D > 0$; then,

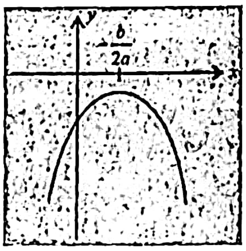


$f(x) > 0\ \forall\, x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) < 0\ \forall\, x \in (\alpha, \beta)$

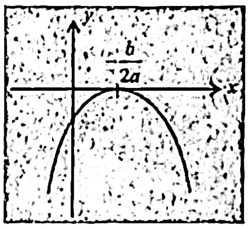
$f(x)_{\min} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$

(*iv*) $a < 0$ and $D < 0 \Leftrightarrow f(x) < 0\ \forall\, x \in R$



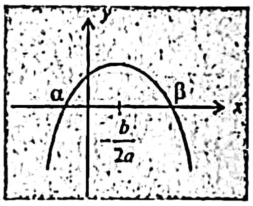
α and β are complex conjugates

(*v*) $a < 0$ and $D = 0 \Leftrightarrow f(x) \leq 0\ \forall\, x \in R$



$f(x) = 0$ at $x = \frac{-b}{2a}$

(*vi*) $a < 0$ and $D > 0$; then



$f(x) < 0\ \forall\, x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) > 0\ \forall\, x \in (\alpha, \beta)$

$f(x)_{\max} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$

Credit:--- Divansh.....

Train Your Brain

Example 25: If $a+b+c=0$; $a\,n^2+b\,n+c=0$ and $a+b\,n+c\,n^2=0$ where $n \neq 0, 1$, then

(*a*) $a=0, b=c=1$

(*b*) $a=b=0, c=1$

(*c*) $a=b=c=0$

(*d*) $a=1, b=c=0$

Sol. (*c*) Note that $a\,x^2+b\,x+c=0$ is satisfied by $x=1$;

$x=n$ & $x=\dfrac{1}{n}$ where $n \neq \dfrac{1}{n}$

⇒ Q.E. has 3 distinct real roots which implies that it must be an identity.

Example 26: If $x=1$ and $x=2$ are solutions of the equation $x^3+ax^2+bx+c=0$ and $a+b=1$, then the value of $b=$

(*a*) 5 (*b*) –4

(*c*) –5 (*d*) 4

Sol. (*a*) $a+b+c=-1$

$\Rightarrow c=-2$ & $8+4a+2b+c=0$

$\Rightarrow 4a+2b=-6$

$\Rightarrow 2a+b=-3$

$\Rightarrow a=-4, b=5$

Example 27: A polynomial in x of degree greater than 3 leaves the remainder 2, 1 and –1 when divided by $(x-1)$, $(x+2)$ & $(x+1)$ respectively, then the remainder, if the polynomial is divided by, $(x^2-1)(x+2)$ is

(*a*) $a+b+c=-1$

(*b*) $a+b+c=-1$

(*c*) $a-b-c=-1$

(*d*) $a-b+c=-1$

Sol. (*d*) $f(x)=Q_1(x-1)+2=Q_2(x+2)+1=Q_3(x+1)-1$

$\Rightarrow f(1)=2\,; f(-2)=1\,; f(-1)=-1$

Let $f(x)=Q_r(x^2-1)(x+2)+ax^2+bx+c$.

Hence $a+b+c=2\,; 4a-2b+c=1$

and $a-b+c=-1$

Example 28: The roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ $(a \neq b \neq c \neq 0)$ are

(*a*) $1, \dfrac{c(a-b)}{a(b-c)}$ (*b*) $1, \dfrac{a(a-b)}{c(b-c)}$

(*c*) $1, \dfrac{a(a+b)}{c(b-c)}$ (*d*) $1, \dfrac{c(a+b)}{a(b+c)}$

Sol. (*a*) $a(b-c)+b(c-a)+c(a-b)=0$

∴ the roots are $1, \dfrac{c(a-b)}{a(b-c)}$

Example 29: The roots of $ax^2+3bx+c=0$ if $3b=a+c$ are

(*a*) $1, \dfrac{c}{a}$ (*b*) $1, \dfrac{c}{-a}$

(*c*) $1, \dfrac{-c}{-a}$ (*d*) $-1, \dfrac{-c}{a}$

Sol. (*d*) $ax^2+3bx+c=0$ and $3b=a+c$

$ax^2+(a+c)x+c=0$; $ax^2+ax+cx+c=0$

$ax(x+1)+c(x+1)=0$; $(x+1)(ax+c)=0$

$x=-1; x=\dfrac{-c}{a}$

∴ roots are $-1, \dfrac{-c}{a}$

Concept Application

17. If p, q are two real numbers satisfying the relations $2p^2-3p-1=0$ and $q^2+3q-2=0$ and $pq \neq 1$, then the value of $\dfrac{pq+p+1}{q}$.

(*a*) 1 (*b*) –1

(*c*) 0 (*d*) None of these

18. If p and q are the roots of $x^2+px+q=0$ then

(*a*) $p=1$ (*b*) $p=1$ or 0

(*c*) $p=-2$ (*d*) $p=-2$ or 0

RANGE OF QUADRATIC EXPRESSION

$f(x)=ax^2+bx+c$

(*i*) Range when x ∈ R : If $a>0$

$$\Rightarrow f(x) \in \left[-\frac{D}{4a}, \infty\right)$$

$$a<0 \Rightarrow f(x) \in \left(-\infty, \frac{-D}{4a}\right]$$

Maximum & Minimum Value of $y=ax^2+bx+c$ occurs at $x=-(b/2a)$ according as $a<0$ or $a>0$ respectively

(*ii*) Range in restricted domain: Given $x \in [x_1, x_2]$

(a) If $-\dfrac{b}{2a} \notin [x_1, x_2]$ then,

$$f(x) \in \left[\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}\right]$$

(b) If $-\dfrac{b}{2a} \in [x_1, x_2]$ then,

$$f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$$

Train Your Brain

Example 30: The minimum value of $f(x) = x^2 - 5x + 6$.

(*a*) $-\frac{1}{3}$ (*b*) $-\frac{1}{2}$

(*c*) $-\frac{1}{4}$ (*d*) None of these

Sol. (*c*) Minimum value of $f(x) = -\frac{D}{4a} = -\left(\frac{25-24}{4}\right)$

$= -\frac{1}{4}\left(\text{at } x = -\frac{b}{2a} = \frac{5}{2}\right)$. Hence range is $\left[-\frac{1}{4}, \infty\right)$.

Example 31: The range of $y = \frac{x+2}{2x^2+3x+6}$, if x is real and $x \neq -2$ is

(*a*) $\left[-\frac{1}{13}, \frac{1}{3}\right]$

(*b*) $\left[-\frac{1}{13}, \frac{1}{3}\right] - \{0\}$

(*c*) $\left[-\frac{1}{13}, \frac{1}{3}\right]$ or $\left[-\frac{1}{13}, \frac{1}{3}\right] - \{0\}$

(*d*) None of these

Sol. (*a*) $y = \frac{x+2}{2x^2+3x+6}$

$\Rightarrow 2yx^2 + 3yx + 6y = x + 2$

$\Rightarrow 2yx^2 + (3y-1)x + 6y - 2 = 0$ ... (*i*)

Case-I: If $y \neq 0$, then equation (*i*) is quadratic in x

$\because$ x is real

$\therefore D \geq 0$

$\Rightarrow (3y-1)^2 - 8y(6y-2) \geq 0 \Rightarrow (3y-1)(13y+1) \leq 0$

$y \in \left[-\frac{1}{13}, \frac{1}{3}\right] - \{0\}$

Case-II: if $y = 0$, then equation becomes

$x = -2$

Which is possible as x is real

$\therefore$ Range $y \in \left[-\frac{1}{13}, \frac{1}{3}\right]$

Example 32: The maximum or minimum values of $2x + 5 - 3x^2$ on R is

(*a*) Maximum value $= \frac{16}{3}$

(*b*) Minimum value $= \frac{16}{3}$

(*c*) Maximum value $= -\frac{16}{3}$

(*d*) Minimum value $= -\frac{16}{3}$

Sol. (*a*) Given expression $2x + 5 - 3x^2$,

$a = -3 < 0, b = 2, c = 5$

$\Rightarrow$ given expression has maximum value

$\therefore$ maximum value $= \frac{4ac-b^2}{4a} = \frac{4(-3)(5)-4}{4(-3)} = \frac{16}{3}$

Concept Application

19. Maximum value of $-5x^2 + 2x + 3$ is

(*a*) $\frac{14}{5}$ (*b*) $\frac{2}{3}$ (*c*) 1 (*d*) $\frac{16}{5}$

20. The maximum value of $\frac{1}{4x^2+2x+1}$ is

(*a*) $\frac{4}{3}$ (*b*) $\frac{2}{3}$ (*c*) 1 (*d*) $\frac{3}{4}$

21. If $x \in [2, 4]$ then for the expression $x^2 - 6x + 5$

(*a*) The least value $= -4$

(*b*) The greatest value $= 4$

(*c*) The least value $= 3$

(*d*) The greatest value $= -5$

22. If $2x - 7 - 5x^2$ has maximum value at $x = a$ then $a =$

(*a*) $\frac{-1}{5}$ (*b*) $\frac{1}{5}$

(*c*) $\frac{34}{5}$ (*d*) $\frac{-34}{5}$

23. Find the range of the following functions:

(*i*) $f(x) = x^2 + x + 1$ (*ii*) $f(x) = 2x^2 + 3x + 5$

(*iii*) $f(x) = x^2 + 2x + 7$ (*iv*) $f(x) = (x-1)(x-2)$

(*v*) $f(x) = \frac{(x-1)}{x^2+x+1}$ (*vi*) $f(x) = \frac{2x^2+3x+1}{x^2+x+1}$

(*vii*) $f(x) = \frac{(x-1)(x-2)}{(x-3)}$ (*viii*) $f(x) = \frac{(x-1)(x-2)}{(x-3)(x-4)}$

(*ix*) $f(x) = \frac{(x-1)(x-3)}{(x-2)(x-4)}$ (*x*) $f(x) = \frac{x^2-5}{2x^2+3}$

(*xi*) $f(x) = x^2 + x + 1; x \geq 0$ (*xii*) $f(x) = 2x^2 + x + 1; 1 \leq x \leq 5$

SOLUTION OF QUADRATIC INEQUALITIES

The values of 'x' satisfying the inequality, $ax^2 + bx + c > 0$ ($a \neq 0$) are:

(*i*) If $D > 0$, i.e. the equation $ax^2 + bx + c = 0$ has two different roots $\alpha < \beta$.

$a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$

$a < 0 \Rightarrow x \in (\alpha, \beta)$

(ii) If $D = 0$, i.e. roots are equal, i.e. $\alpha = \beta$.

$a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\alpha, \infty)$

$a < 0 \Rightarrow x \in \phi$

(iii) If $D < 0$, i.e. the equation $ax^2 + bx + c = 0$ has no real roots.

$a > 0 \Rightarrow x \in R$

$a < 0 \Rightarrow x \in \phi$

(iv) Inequalities of the form $\dfrac{P(x)}{Q(x)} \lesseqgtr 0$ can be solved using the method of intervals.

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Example 33: The solution set of k so that $y = kx$ is secant to the curve $y = x^2 + k$ is

(a) $(-\infty, 0) \cup (4, \infty)$ (b) $(-\infty, 0] \cup (4, \infty)$

(c) $(-\infty, 0) \cup [4, \infty)$ (d) $(-\infty, 0] \cup [4, \infty)$

Sol. (a) Put $y = kx$ in $y = x^2 + k$

$\Rightarrow kx = x^2 + k = 0$

$\Rightarrow x^2 - kx + k = 0$ for line to be secant, $D > 0$

$\Rightarrow k^2 - 4k > 0\ k(k-4) > 0$

hence $k > 4$ or $k < 0$

$\Rightarrow k \in (-\infty, 0) \cup (4, \infty)$

Example 34: The solutions set of inequality $\dfrac{x^2+x+1}{|x+1|} > 0$ is

(a) R (b) $R - \{1\}$

(c) $R - \{-1\}$ (d) None of these

Sol. (c) $\dfrac{x^2+x+1}{|x+1|} > 0$ ... (i)

$\therefore\ |x+1| > 0, \forall x \in R - \{-1\}$

$\therefore$ (i) becomes $x^2 + x + 1 > 0\ x \neq -1$

$\because\ x^2 + x + 1 > 0, \forall x \in R$ (as its $a > 0$ and $D < 0$)

$\therefore$ Solution of (i) is $x \in R - \{-1\}$

Example 35: The solutions set of inequality $\left|\dfrac{x^2-3x-1}{x^2+x+1}\right| < 3$ is.

(a) $(-\infty, -2) \cup (-1, \infty)$ (b) $(-\infty, -2] \cup (-1, \infty)$

(c) $(-\infty, -2] \cup [-1, \infty)$ (d) $(-\infty, -2) \cup [-1, \infty)$

Sol. (a) $\left|\dfrac{x^2-3x-1}{x^2+x+1}\right| < 3$.

$\because\ (x^2 + x + 1) > 0, \forall x \in R$

$\therefore\ |x^2 - 3x - 1| < 3(x^2 + x + 1)$

$\Rightarrow (x^2 - 3x - 1)^2 - \{3(x^2 + x + 1)\}^2 < 0$

$\Rightarrow (4x^2 + 2)(-2x^2 - 6x - 4) < 0$

$\Rightarrow (2x^2 + 1)(x + 2)(x + 1) > 0$

$\Rightarrow x \in (-\infty, -2) \cup (-1, \infty)$

Concept Application

24. The set of values of x for which the inequality $[x]^2 - 5[x] + 6 \leq 0$ (where [.] denote the greatest function) hold good is

(a) $2 \leq [x] < 3$ (b) $2 \leq x < 4$

(c) $2 \leq x < 3$ (d) $2 \leq x \leq 4$

25. Let $f(x) = ax^2 + bx + c$, if $f(-1) < 1, f(1) > -1, f(3) < -4$ and $a \neq 0$ then

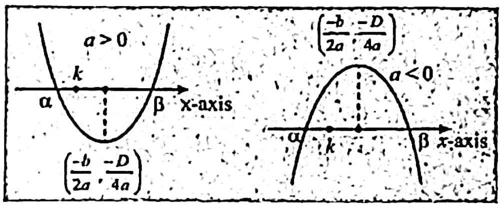
(a) $a > 0$ (b) $a > 1$

(c) $a < -\dfrac{1}{8}$ (d) $-\dfrac{1}{8} < a < 0$

LOCATION OF ROOTS

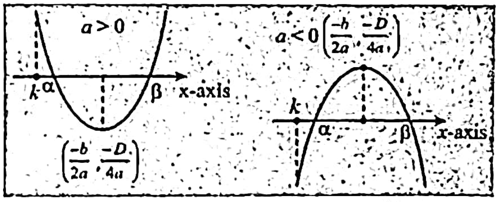
Let $f(x) = ax^2 + bx + c, a, b, c \in R, a \neq 0$ and α, β be roots of $f(x) = 0$

1. A real number k lies between the roots of $f(x) = 0$


$a > 0$
k
α
β
x-axis
$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$
$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$
$a < 0$
α
k
β
x-axis

(i) $D > 0$, (ii) $af(k) < 0$, where $\alpha < \beta$

2. If both roots of quadratic equation $f(x) = 0$ are greater than k, then

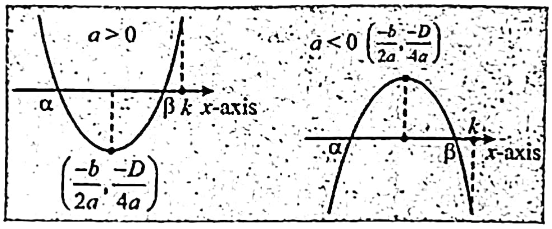

$a > 0$
k
α
β
x-axis
$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$
$a < 0$ $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$
k
α
β
x-axis

(i) $D \geq 0$,

(ii) $af(k) > 0$,

(iii) $k < \dfrac{-b}{2a}$, where $\alpha \leq \beta$

3. If both roots are less than real number k, then


$a > 0$
α
β k x-axis
$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$
$a < 0$ $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$
k
α
β
x-axis

(i) $D \geq 0$,

(ii) $af(k) > 0$

(iii) $k > \dfrac{-b}{2a}$, where $\alpha \leq \beta$

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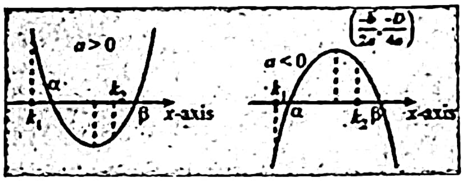
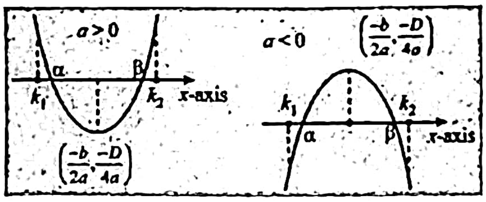
4. Exactly one root lies between real numbers k_1 and k_2, where $k_1 < k_2$.



(i) $D > 0$,

(ii) $f(k_1).f(k_2) < 0$, where $\alpha < \beta$

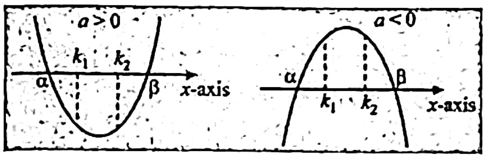
5. If both roots of $f(x) = 0$ are confined between real numbers k_1 and k_2, where $k_1 < k_2$.


a > 0
α
β
k_1
k_2
x-axis
$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$
a < 0
$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$
k_1
k_2
α
β
x-axis

(i) $D \ge 0$, (ii) $af(k_1) > 0$

(iii) $af(k_2) > 0$, (iv) $k_1 < \frac{-b}{2a} < k_2$, where $\alpha \le \beta$

6. If real numbers k_1 and k_2 lie between roots of $f(x) = 0$, where $k_1 < k_2$.



(i) $D > 0$,

(ii) $af(k_1) < 0$

(iii) $af(k_2) < 0$, where $\alpha < \beta$

Train Your Brain

Example 36: Given that α is a real number not less than −1, such that the equation in x i.e., $x^2 + 2(\alpha - 2)x + \alpha^2 - 3\alpha + 3 = 0$, then the maximum value of $\frac{\alpha x_1^2}{1-x_1} + \frac{\alpha x_2^2}{1-x_2}$ is

(a) 10 (b) 5 (c) −10 (d) −5

Sol. (a) $\frac{\alpha x_1^2}{1-x_1} + \frac{\alpha x_2^2}{1-x_2} = \frac{\alpha\left[x_1^2(1-x_2) + x_2^2(1-x_1)\right]}{(1-x_1)(1-x_2)}$

$$= \frac{\alpha\left[x_1^2 + x_2^2 - x_1x_2(x_1 + x_2)\right]}{x_1x_2 - (x_1 + x_2) + 1}$$

$$= \frac{\alpha[(2\alpha^2 - 10\alpha + 10) + 2(\alpha^2 - 3\alpha + 3)(\alpha - 2)]}{\alpha^2 - 3\alpha + 3 + 2(\alpha - 2) + 1}$$

$$= \frac{\alpha(2\alpha^3 - 8\alpha^2 + 8\alpha - 2)}{\alpha^2 - \alpha} = 2(\alpha^2 - 3\alpha + 1)$$

$$= 2\left(\alpha - \frac{3}{2}\right)^2 - \frac{5}{2} \le 2\left(-1 - \frac{3}{2}\right)^2 - \frac{5}{2} = 10 \text{ since } -1 \le \alpha < 1.$$

Thus, the maximum value of $\frac{\alpha x_1^2}{1-x_1} + \frac{\alpha x_2^2}{1-x_2}$ is 10.

Example 37: The range of values of 'a' for which the equation $x^2 - (a-5)x + \left(a - \frac{15}{4}\right) = 0$ has at least one positive root.

(a) $(-\infty, 15/4) \cup [10, \infty)$

(b) $(-\infty, 15/4) \cup (10, \infty)$

(c) $(-\infty, 15/4] \cup [10, \infty)$

(d) None of these

Sol. (c) $D \ge 0 \Rightarrow (a-5)^2 - 4\left(a - \frac{15}{4}\right) \ge 0$

$\Rightarrow a^2 - 10a + 25 - 4a + 15 \ge 0$

$\Rightarrow a^2 - 14a + 40 \ge 0$

$\Rightarrow (a-4)(a-10) \ge 0$

$\Rightarrow a \in (-\infty, 4] \cup [10, \infty)$

Case I: When both roots are positive

$D \ge 0, a - 5 > 0, a - \frac{15}{4} > 0$

$\Rightarrow D \ge 0, a > 5, a > \frac{15}{4}$

$\Rightarrow a \in [10, \infty)$

Case II: When exactly one root is positive

$\Rightarrow a - \frac{15}{4} \le 0$

$\Rightarrow a \le \frac{15}{4}$

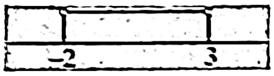
then finally $a \in (-\infty, 15/4] \cup [10, \infty)$

Example 38: The set of values of k for which one root of the equation of $x^2 - (k+1)x + k^2 + k - 8 = 0$ exceed 2 and other is smaller than 2 is

(a) $(-2, 3)$ (b) $[-2, 3)$

(c) $[-2, 3]$ (d) $(2, -3)$

Sol. (a) Since $a > 0$, $\therefore f(2) < 0$



$\therefore 4 - 2(k+1) + k^2 + k - 8 < 0$

$\Rightarrow k^2 - k + 6 < 0$

$\Rightarrow (k+2)(k-3) < 0 \quad \therefore k \in (-2, 3)$

Concept Application

26. Let $x^2 - (m-3)x + m = 0$ $(m \in R)$ be a quadratic equation, then find the values of 'm' for which
    (*a*) Both the roots are greater than 2.
    (*b*) Both roots are positive.
    (*c*) One root is positive and other is negative.
    (*d*) One root is greater than 2 and other smaller than 1
    (*e*) Roots are equal in magnitude and opposite in sign.
    (*f*) Both roots lie in the interval (1, 2)

27. If both roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then the interval in which k lies
    (*a*) $(-\infty, 4)$ (*b*) $(-\infty, 4]$
    (*c*) $(-\infty, -4)$ (*d*) $(-\infty, -4]$

28. The interval in which 'a' lies when the graph of the function $y = 16x^2 + 8(a+5) - 7a - 5$ is strictly above the x-axis is
    (*a*) $-15 < a < 2$ (*b*) $-15 < a < 3$
    (*c*) $-15 < a < -2$ (*d*) None of these

NEWTON'S THEOREM

Let α, β be the roots of quadratic equation $ax^2 + bx + c = 0$ and ${}_k = \alpha^k + \beta^k$ then $as_3 + bs_2 + cs_1 = 0$

DESCARTE'S RULE OF SIGNS

The poynomial equations $f(x) = 0$ with real coefficent cannot have more.

(*i*) Positive roots than the number of changes of sign in $f(x)$.

(*ii*) Negative roots than the number of changes of signs in $f(x)$.

ADVANCED LEARNING

Equations Reducible to Quadratic Equations

1. **Reciprocal equation of the standard form can be reduced to an equation of half its dimension.**
2. **Equations of the form**
   (a) $(x-a)(x-b)(x-c)(x-d) = A$ where $a<b<c<d$ can be solved by change of variable.
   $$y = x - \left(\frac{a+b+c+d}{4}\right)$$
   (b) $(x-a)(x-b)(x-c)(x-d) = Ax^2$ where $ab = cd$ can be solved by assumption $y = x + \frac{ab}{x}$.
3. **For the equation of the type $(x-a)^4 + (x-b)^4 = A$.**
   Substitute $y = \dfrac{(x-a)+(x-b)}{2}$

ADVANCED LEARNING

Important Theorems and Results

1. **Every equation of an even degree whose last term is negative has at least two real roots, one positive and one negative, provided that the coefficient of the first term is positive.**
2. **Every equation of nth degree $(n \geq 1)$ has exactly n roots & if the equation has more than n roots, it is an identity.**
3. If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs.**
4. If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.
5. If there be any two real numbers 'a' & 'b' such that $f(a)$ & $f(b)$ are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b'.
6. If an equation has no odd powers of x, then all roots of the equation are complex provided all the coefficients of the equation are having positive sign.
7. If $x = \alpha$ is root repeated m times in $f(x) = 0$,
   ($f(x) = 0$ is an n^{th} degree equation in x) then
   $f(x) = (x-\alpha)^m g(x)$
   when $g(x)$ is a polynomial of degree $(n-m)$ and the root $x = \alpha$ is repeated $(m-1)$ times in $f'(x) = 0$, $(m-2)$ times in $f''(x) = 0, \ldots (m-(m-1))$ times in $f^{m-1}(x) = 0$.
8. The condition that a quadratic function
   $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factor if
   $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$
   or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is 0.

Train Your Brain

Example 39: The solutions of equation $2x^4 + x^3 - 11x^2 + x + 2 = 0$ is

(*a*) $\frac{1}{2}, 2, \frac{3 \pm \sqrt{5}}{2}$ (*b*) $\frac{1}{2}, 2, \frac{-3 \pm \sqrt{5}}{2}$

(*c*) $\frac{1}{2}, 2, \frac{3 \pm \sqrt{5}}{-2}$ (*d*) None of these

Sol. (*b*) Since $x = 0$ is not a solution hence, divide by x^2

We get,

$$2x^2 + x - 11 + \frac{1}{x} + \frac{2}{x^2} = 0$$

Credit:--- Divansh.....

or $2\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-11=0$

Let $x+\frac{1}{x}=y$

$\Rightarrow 2(y^2-2)+y-11=0$

or $2y^2+y-15=0 \Rightarrow y=-3,\frac{5}{2}$

Corresponding values of x are $\frac{1}{2},2,\frac{-3\pm\sqrt{5}}{2}$

Example 40: The value of $\sqrt{7+\sqrt{7-\sqrt{7+\sqrt{7}\ldots\infty}}}$ is

(*a*) 5 (*b*) 4 (*c*) 3 (*d*) 2

Sol. (*c*) Let $\sqrt{7+\sqrt{7-\sqrt{7+\sqrt{7}\ldots\infty}}}$ or $y=\sqrt{7+\sqrt{7-y}}$

or $y^2-7=\sqrt{7-y}$ ... (*i*)

Clearly $y^2-7\geq 0 \Rightarrow y^2\geq 7$

$\Rightarrow y\geq\sqrt{7}$ or $y\leq-\sqrt{7}$

$y\leq-\sqrt{7}$ is not acceptable as $y\geq 0$ so only solution may be in the interval $y\geq\sqrt{7}$

Since $\sqrt{7}>2$ so $y=2$ is not possible

Keeping $y=3$, 4 and 5 in equation (*i*) we see

That only $y=3$ satisfies it and not $y=4$ or $y=5$.

So only solution is $y=3$.

Example 41: If $0\leq x\leq\pi$, then the solution of the equation $16^{\sin^2 x}+16^{\cos^2 x}=10$ is given by

(*a*) $\frac{\pi}{6},\frac{\pi}{3}$ (*b*) $\frac{\pi}{3},\frac{\pi}{2}$

(*c*) $\frac{\pi}{6},\frac{\pi}{2}$ (*d*) None of these

Sol. (*a*) Let $16^{\sin^2 x}=y$, then $16^{\cos^2 x}=16^{1-\sin^2 x}=\frac{16}{y}$

Hence, $y+\frac{16}{y}=10 \Rightarrow y^2-10y+16=0$ or $y=2, 8$

Now $16^{\sin^2 x}=2 \Rightarrow 4\sin^2 x=1$

$\therefore \sin x=\pm\frac{1}{2}\Rightarrow x=\frac{\pi}{6}$

Also $16^{\sin^2 x}=8 \Rightarrow 4\sin^2 x=3$

$\therefore \sin x=\pm\frac{\sqrt{3}}{2}\Rightarrow x=\frac{\pi}{3}$

Hence (*a*) is the correct answer.

Example 42: Given that the quadratic equation $ax^2+bx+c=0$ has no real roots, but Kishan got two roots 2 and 4 since he wrote down a wrong value of a. Rahul also got two roots -1 and 4 because he wrote down the sign of a term wrongly, then the value of $\frac{2b+3c}{a}=$

(*a*) -6 (*b*) 5

(*c*) 6 (*d*) -5

Sol. (*c*) Suppose that Kishan wrote down the coefficient a as a' wrongly. Then

$-\frac{b}{a'}=6,\ \frac{c}{a'}=8$, therefore $-\frac{b}{c}=\frac{3}{4}$. Based on Rahul result, we have $\left|\frac{b}{a}\right|=3,\ \left|\frac{c}{a}\right|=4.$

Since $\Delta=b^2-4ac<0$, we have $ac>0$, therefore $\frac{c}{a}=4$, and hence $\frac{b}{a}=-3$.

Thus, $\frac{2b+3c}{a}=2(-3)+3(4)=6.$

Concept Application

29. The condition for which the equation $ax^2+2cxy+by^2+2bx+2ay+c$ is resolvable into two linear factors is
    (*a*) $a^3+b^3+c^3=3abc$
    (*b*) $a^3+b^3+c^3=abc$
    (*c*) $a^3+b^3+c^3=ab+bc+ca$
    (*d*) $a^3+b^3+c^3+abc=0$
30. If $x^4+4xy+4y^2+4x+cy+3$ can be written as the product of two linear factors, then $c=$
    (*a*) 2 (*b*) 4 (*c*) 6 (*d*) 8
31. The solution set of $(x+4)(x+6)(x+8)(x+12)=1680x^2$ is
    (*a*) $\{2,24,-28-4\sqrt{46},-28+4\sqrt{46}\}$
    (*b*) $\{2,28,-28-4\sqrt{46},-28+4\sqrt{46}\}$
    (*c*) $\{4,28,-28-4\sqrt{46},-28+4\sqrt{46}\}$
    (*d*) None of these
32. If $\left(7+4\sqrt{3}\right)^{x^2-8}+\left(7-4\sqrt{3}\right)^{x^2-8}=14$ then the value of x is
    (*a*) $x=\pm\sqrt{7}, x=\pm 3$ (*b*) $x=\sqrt{7}, x=\pm 3$
    (*c*) $x=\pm\sqrt{7}, x=3$ (*d*) $x=\sqrt{7}, x=3$

Credit:--- Divansh.....

Aarambh (Solved Examples)

1. Let a, b and c be real numbers such that $4a + 2b + c = 0$ and $ab > 0$. Then the equation $ax^2 + bx + c = 0$ has
   (*a*) real roots (*b*) imaginary roots
   (*c*) exactly one root (*d*) none of these

Sol. $D = b^2 - 4ac = b^2 - 4a(-4a - 2b) = b^2 + 16a^2 + 8ab$

Since $ab > 0$

$\therefore D > 0$

So equation has real roots.

Therefore, option (*a*) is the correct answer.

2. The quadratic equation whose roots are the x and y intercepts of the line passing through (1, 1) and making a triangle of area A with the axes, may be
   (*a*) $x^2 + Ax + 2A = 0$ (*b*) $x^2 - 2Ax + 2A = 0$
   (*c*) $x^2 - Ax + 2A = 0$ (*d*) $x^2 - 2Ax - 2A = 0$

Sol. Line passing through (1, 1) is

$y - 1 = m(x - 1)$

$y = mx + 1 - m$

x intercept $x = \dfrac{m-1}{m}$ ... (*i*)

$y = 0$

y intercept : $y = 1 - m$ ... (*ii*)

$x = 0$

Given area of $OAB = A = \frac{1}{2} \times (OB)(OA)$

$= \frac{1}{2} \times (1-m)\frac{(m-1)}{m}$

$A = \dfrac{-(m^2-1)}{2m}$ ... (*iii*)

Quadratic equation whose roots are x, y, is

$x_1^2 - (x+y)x_1 + xy = 0$

$$x_1^2 - \left(\frac{2m-m^2-1}{m}\right)x_1 - \frac{(m-1)^2}{m} = 0$$

$$x_1^2 + \left(\frac{m^2-2m+1}{m}\right)x_1 - \frac{(m-1)^2}{m} = 0$$

From (*iii*), we get

$x_1^2 - 2Ax_1 + 2A = 0$

Therefore, option (*b*) is the correct answer.

3. Given that the equation $x^2 + (2a-1)x + a^2 = 0$ has two real positive roots, where a is an integer. If x_1 and x_2 are the roots, find the value of $\left|\sqrt{x_1} - \sqrt{x_2}\right|$.
   (*a*) 1 (*b*) −1
   (*c*) 2 (*d*) None of these

Sol. The two real roots are positive implies $1 - 2a \geq 0$, i.e., since a is an integer, so $a \leq 0$. Then

$$\left|\sqrt{x_1} - \sqrt{x_2}\right| = \sqrt{\left(\sqrt{x_1} - \sqrt{x_2}\right)^2} = \sqrt{x_1 + x_2 - 2\sqrt{x_1x_2}}$$

$= \sqrt{1 - 2a + 2a} = 1.$

Therefore, option (*a*) is the correct answer.

4. If α & β are the two distinct roots of $x^2 + 2(k-3)x + 9 = 0$, then the values of k such that $\alpha, \beta \in (-6, 1)$ are
   (*a*) $\left[6, \frac{27}{4}\right)$ (*b*) $\left(6, \frac{27}{4}\right]$ (*c*) $\left(6, \frac{27}{4}\right)$ (*d*) $\left[6, \frac{27}{4}\right]$

Sol. $x^2 + 2(k-3)x + 9 = 0$ ... (*i*)

Roots α, β of equation (*i*) are distinct and lies between − 6 and 1

$D > 0$

$\Rightarrow \quad 4(k-3)^2 - 36 > 0 \Rightarrow k(k-6) > 0$

$\Rightarrow \quad k \in (-\infty, 0) \cup (6, \infty)$ ... (*ii*)

$f(1) > 0$

$\Rightarrow \quad 1 + 2(k-3) + 9 > 0 \quad \Rightarrow k \in (-2, \infty)$ ... (*iii*)

$f(-6) > 0 \quad \Rightarrow 36 - 12(k-3) + 9 > 0$

$\Rightarrow \quad 4k - 27 < 0 \quad \Rightarrow k \in \left(-\infty, \frac{27}{4}\right)$ ... (*iv*)

$$-6 < -\frac{b}{2a} < 1 \Rightarrow -6 < \frac{-2(k-3)}{2} < 1$$

$\Rightarrow -1 < k - 3 < 6 \Rightarrow 2 < k < 9$ ... (*v*)

$(ii) \cap (iii) \cap (iv) \cap (v) \Rightarrow k \in \left(6, \frac{27}{4}\right)$

Therefore, option (*c*) is the correct answer.

5. If α is a root of the equation $4x^2 + 2x - 1 = 0$ then the other root must be
   (*a*) $3\alpha^3 + 4\alpha$ (*b*) $4\alpha^3 + 3\alpha$
   (*c*) $2\alpha^2 - 1$ (*d*) All of the above

Sol. $4x^2 + 2x - 1 = 0$

$$x = \frac{-2 \pm \sqrt{4+16}}{8}$$

$$x = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

If $\alpha = \dfrac{\sqrt{5}-1}{4} = \sin 18°$

$\beta = -\left[\dfrac{\sqrt{5}+1}{4}\right] = -\cos 36°$

$\beta = -\cos 36° = -\sin[90° - 36°] = -\sin 54°$

$= 4\sin^3 18° - 3\sin 18° = 4\alpha^3 - 3\alpha$

$\beta = -\cos 36° = 2\sin^2 18° - 1 = 2\alpha^2 - 1$

Therefore, option (*c*) is the correct answer.

6. Given that a, b, c are the lengths of three sides of ΔABC, $a > b > c$, $2b = a + c$, and b is a positive integer. If $a^2 + b^2 + c^2 = 84$, then the value of b =
   (*a*) 5 (*b*) −5
   (*c*) 4 (*d*) None of these

Sol. The conditions $a + c = 2b$ and $a^2 + b^2 + c^2 = 84$ yield

$$ac = \frac{1}{2}\left[(a+c)^2 - (a^2+c^2)\right] = \frac{1}{2}(5b^2 - 84)$$

By the inverse Viete throrem, the equation $x^2 - 2bx + \dfrac{5b^2 - 84}{2} = 0$ has two distinct roots a and c, so its discriminant is positive, i.e., $\Delta = 4b^2 - 2(5b^2 - 84) = 168 - 6b^2 > 0$, which implies

Credit:--- Divansh.....

$b^2 < \frac{168}{6} = 28$

Since $ac > 0$ implies $84 < 5b^2$, so $16 < \frac{84}{5} < b^2$. Thus, $16 < b^2 < 28$ yields $b = 5$

Therefore, option (*a*) is the correct answer.

7. $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$. Solve for x.

(*a*) (5, 10) (*b*) [1, ∞) (*c*) [5, 10] (*d*) (5, 10)

Sol. Put $x - 1 = t^2$ in the given equation, we get

$$\sqrt{t^2+1+3-4t}+\sqrt{t^2+1+8-6t}=1$$

$$\Rightarrow |t-2|+|t-3|=1$$

$$\Rightarrow t\in[2,3] \Rightarrow x\in[5,10]$$

Therefore, option (*c*) is the correct answer.

8. If the roots of the quadratic equation $x^2 + mx + n = 0$ are tan 30° and tan 15°, respectively, then the value of $2 + n - m$ =

(*a*) 3 (*b*) −3 (*c*) −4 (*d*) 4

Sol. The equation $x^2 + mx + n = 0$ has roots tan 30° and tan 15°.

Therefore,

$\tan 30° + \tan 15° = -m$ ... (*i*)

$\tan 30° \tan 15° = n$ ... (*ii*)

Now, $\tan 45° = \tan(30° + 15°) \Rightarrow 1 = \frac{\tan 30° + \tan 15°}{1 - \tan 30° \tan 15°}$

$\Rightarrow 1 = \frac{-m}{1-n}$ [Using (*i*) and (*ii*)]

$\Rightarrow 1 - n = -m \quad \Rightarrow n - m = 1 \Rightarrow 2 + n - m = 3$

Therefore, option (*a*) is the correct answer.

9. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a are

(*a*) (−3, 3) (*b*) [−3, 3) (*c*) (−3, 3] (*d*) [−3, 3]

Sol. If α, β are roots of $x^2 + ax + 1 = 0$, then

$$|\alpha-\beta|<\sqrt{5} \Rightarrow \left|\sqrt{(\alpha+\beta)^2-4\alpha\beta}\right|<\sqrt{5}$$

$$\Rightarrow \left|\sqrt{a^2-4}\right|<\sqrt{5} \Rightarrow a^2-4<5 \Rightarrow a^2-9<0$$

$\therefore a \in (-3, 3)$

Therefore, option (*a*) is the correct answer.

10. The values of a for which one root of equation $(a - 5)x^2 - 2ax + a - 4 = 0$ is smaller than 1 and the other greater than 2 are

(*a*) [5, 24) (*b*) (5, 24) (*c*) (5, 24] (*d*) [5, 24]

Sol. Let $f(x) = (a - 5)x^2 - 2ax + a - 4$ $(a \neq 5)$ as 1 and 2 lie between the roots of $f(x) = 0$

Consider $(a - 5)f(1) < 0$: $(a - 5)(a - 5 - 2a + a - 4) < 0$

$\Rightarrow (a - 5)(-9) < 0 \Rightarrow a - 5 > 0 \Rightarrow a \in (5, \infty)$ ... (*i*)

Consider $(a - 5)f(2) < 0$: $(a - 5)(4(a - 5) - 4a + a - 4) < 0$

$\Rightarrow (a - 5)(a - 24) < 0 \Rightarrow a \in (5, 24)$ ... (*ii*)

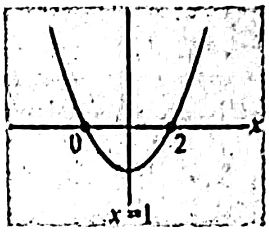
Hence, the values of a satisfying (*i*) and (*ii*) at the same time are $a \in (5, 24)$

Therefore, option (*b*) is the correct answer.

11. Let $f(x) = ax^2 + bx + c$ be a polynomial. If $f(1 - x) = f(1 + x)$, $f(2) = 0$ and $f(3) = 6$, then the value of $a + b + c$ =

(*a*) −2 (*b*) 2 (*c*) −4 (*d*) 4

Sol. $f(1 - x) = f(1 + x)$



⇒ the graph of the function $y = f(x)$ will be symmetric about the line $x = 1$

$f(2) = 0 \Rightarrow f(0) = 0$

$f(x) = ax(x - 2)$

Now $f(3) = 6 \Rightarrow a = 2$

$f(x) = 2x^2 - 4x$

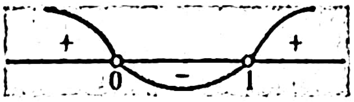
Therefore, option (*a*) is the correct answer.

12. All values of a for which the equation $2x^2 - 2(2a + 1)x + a(a + 1) = 0$ has two roots, one of which is greater than a and the other is smaller than a.

(*a*) (−∞, −1] ∪ (0, ∞) (*b*) (−∞, −1) ∪ [0, ∞)

(*c*) (−∞, −1) ∪ (0, ∞) (*d*) (−∞, −1] ∪ [0, ∞)

Sol. Here coefficient of x^2 is positive



Let $f(x) = 2x^2 - 2(2a + 1)x + a(a + 1)$

$\therefore f(a) = 2a^2 - 2(2a + 1)a + a(a + 1) = -a^2 - a < 0$

$\Rightarrow -a(a + 1) < 0 \Rightarrow a(a + 1) > 0$

$a \in (-\infty, -1) \cup (0, \infty)$

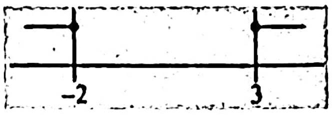
Therefore, option (*c*) is the correct answer.

13. If x is real and $4y^2 + 4xy + x + 6 = 0$, then the complete set of values of x for which y is real is,

(*a*) (−∞, −2] ∪ [3, ∞) (*b*) (−∞, −2] ∪ (3, ∞)

(*c*) (−∞, −2) ∪ [3, ∞) (*d*) (−∞, −2) ∪ (3, ∞)

Sol. $4y^2 + 4xy + x + 6 = 0 \quad y \in R$



$\therefore D \geq 0$

$16x^2 - 16(x + 6) \geq 0$

$\Rightarrow x^2 - x - 6 \geq 0$

$\Rightarrow (x - 3)(x + 2) \geq 0$

$\therefore x \in (-\infty, -2] \cup [3, \infty)$

Therefore, option (*a*) is the correct answer.

14. If $Q_1(x) = x^2 + (k - 29)x - k$ and $Q_2(x) = 2x^2 + (2k - 43)x + k$ both are factors of a cubic polynomial $P(x)$, then the largest value of k is

(*a*) 0 (*b*) 33 (*c*) 23 (*d*) 30

Sol. Two quadratic polynomials can be a factor of cubic polynomial only when they have atleast one root common

$\Rightarrow x^2 + (k - 29)x - k = 0$ ... (*i*)

and $2x^2 + (2k - 43)x + k = 0$ ... (*ii*)

Must have a common roots

Multiple equation (*i*) by 2 and subtracting, we get

$15x + 3k = 0 \Rightarrow x = \frac{-k}{5}$ is the common root and it must satisfy equation (i)

$$\Rightarrow \frac{k^2}{25} + (k-29)\left(-\frac{k}{5}\right) - k = 0$$

$$\Rightarrow \left(-\frac{k}{5}\right)\left[-\frac{k}{5} + k - 29 + 5\right] = 0 \quad \Rightarrow \quad k = 0 \text{ or } k = 30$$

Therefore, option (*d*) is the correct answer.

Credit:--- Divansh.....

School Level Problems

SINGLE CORRECT TYPE QUESTION

1. The roots of the quadratic equation $x+\frac{1}{x}=3, x\neq 0$ are

   (a) $3+\sqrt{5}, 3-\sqrt{5}$ (b) $2+\sqrt{5}, 2-\sqrt{5}$

   (c) $\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$ (d) $\frac{3+\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}$

2. If the roots of the equation $x^2+2x+p=0$ are real then the value of P is

   (a) $P\leq 2$ (b) $P\leq 1$

   (c) $P\leq 3$ (d) None of these

3. If the roots of $ax^2+bx+c=0$ are in the ratio $m:n$, then

   (a) $mna^2=(m+n)c^2$ (b) $mnb^2=(m+n)ac$

   (c) $mnb^2=(m+n)^2ac$ (d) $mnb^2=(m-n)^2ac$

4. If one root of the equation $x^2+px+12=0$ is 4, while the equation $x^2+px+q=0$ has equal roots, the value of q is

   (a) $\frac{49}{4}$ (b) $\frac{4}{49}$ (c) 4 (d) 49

5. The roots of the quadratic equation

   $$\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}, a+b\neq 0$$

   (a) a, b (b) $-a, b$ (c) $a, -b$ (d) $-a, -b$

6. The roots of $x^2-2x-(r^2-1)=0$ are:

   (a) $1-r, r-1$ (b) $1-r, r+1$

   (c) $1, r$ (d) $1-r, r$

7. If equation $\frac{x^2-bx}{ax-c}=\frac{k-1}{k+1}$ has equal and opposite roots in sign then the value of k is

   (a) $\frac{a+b}{a-b}$ (b) $\frac{a-b}{a+b}$ (c) $\frac{a}{b}+1$ (d) $\frac{a}{b}-1$

8. If the equation $x^2-bx+1=0$ does not possess real roots, then

   (a) $-3<b<3$ (b) $-2<b<2$

   (c) $b>2$ (d) $b<-2$

9. The roots of the equation $x^2-2\sqrt{2}x+1=0$ are

   (a) Real and different (b) Imaginary and different

   (c) Real and equal (d) Rational and different

VERY SHORT ANSWER TYPE QUESTIONS

10. Find the quadratic equation with rational coefficients when one root is $\frac{1}{2+\sqrt{5}}$.

11. Find the values of k for which exactly one root of the quadratic equation $4x^2-4(k-2)x+k-2=0$ lies in $\left(0,\frac{1}{2}\right)$.

12. Find the zeroes of a biquadratic equation $x^4-3x^2+2=0$.

13. Find the range of k for which 6 lies between the roots of the quadratic equation $x^2+2(k-3)x+9=0$.

14. Find the values of x for which the expression $\frac{x^2-4x+3}{x^2+x+1}<0$.

    Find the minimum and maximum values of quadratic expression $f(x)=x^2-12x+11$.

15. Find the relation between c, a and m if a quadratic equation formed from $y^2=4ax$ and $y=mx+c$ has equal roots.

16. Check whether the following is a quadratic equation $(x+2)^3=x^3-4$

LONG ANSWER TYPE QUESTIONS

17. Find the values of k such that the equation $\frac{p}{(x+r)}+\frac{q}{(x-r)}=\frac{k}{2x}$ has two equal roots.

18. For what value of k do both the quadratic equations, $6x^2-17x+12=0$ and $3x^2-2x+k=0$, will have a common root?

19. For what values of 'm' does the quadratic expression $x^2+2(m+1)x+9m-5=0$ have only negative roots?

20 Find the values of k for which the roots of the quadratic equation $x^2-(k-3)x+k=0$ are greater than 2.

21. Find the range of function $f(x)=\frac{(x+2)}{(2x^2+3x+6)}$, if x is real.

CASE STUDY BASED QUESTIONS

Case Study-I

Nidhi and Ria are very close friends. Nidhi's parents own a Maruti Alto. Ria's parents own a Toyota Liva. Both the families decide to go for a picnic to Somnath temple in Gujrat by their own cars.

Nidhi's car travels x km/h while Ria's car travels 5 km/h more than Nidhi's car. Nidhi's car took 4 hrs more than Ria's car in covering 400 km.

22. Write the quadratic equation that describe the speed of Nidhi's car?

    (a) $x^2+5x-500=0$ (b) $x^2+4x-200=0$

    (c) $x^2+6x-300=0$ (d) $x^2+5x+500=0$

23. How much time did Ria take to travel 400 km?

    (a) 18 hours (b) 16 hours

    (c) 12 hours (d) 11 hours

Case Study-II

Quadratic equations started around 3000 B.C. with the Babylonians. They were one of the world's first civilisation, and came up with some great ideas like agriculture, irrigation and writing. There were many reasons why Babylonians needed to solve quadratic equations. For example to know what amount of crop you can grow on the square field.

Based on the above information, represent the following questions in the form of quadratic equation.

24. The sum of squares of two consecutive integers is 650.
    (*a*) $x^3 - 2x - 650 = 0$
    (*b*) $2x^2 + 2x - 649 = 0$
    (*c*) $x^3 - 2x^2 - 650 = 0$
    (*d*) $2x^2 + 6x - 550 = 0$

25. A natural number whose square diminished by 84 is thrice of 8 more of given number.
    (*a*) $x^2 + 8x - 84 = 0$ (*b*) $3x^2 - 84x + 3 = 0$
    (*c*) $x^2 - 3x - 108 = 0$ (*d*) $x^2 - 11x + 60 = 0$

Prarambh (Topicwise)

QUADRATIC EQUATION & NATURE OF ROOTS

1. If $a + b + c = 0$, and $a, b, c \in R$, then the roots of the equation $4ax^2 + 3bx + 2c = 0$ are
   (*a*) Equal (*b*) Imaginary
   (*c*) Real (*d*) Both (*a*) and (*b*)

2. The roots of the given equation $2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$ are
   [Where $a \neq b$]
   (*a*) Rational (*b*) Irrational
   (*c*) Real (*d*) Imaginary

3. If $a, b, c \in Q$, then roots of the equation
   $(b + c - 2a)x^2 + (c + a - 2b)x + (a + b - 2c) = 0$ are
   (*a*) Rational (*b*) Non-real
   (*c*) Irrational (*d*) Equal

4. The value of k for which the quadratic equation, $kx^2 + 1 = kx + 3x - 11x^2$ has real and equal roots are
   (*a*) $-11, -3$ (*b*) $5, 7$
   (*c*) $5, -7$ (*d*) $-7, 25$

5. If $x^2 - 3x + 2$ be a factor of $x^4 - px^2 + q$, then $(p, q) =$
   (*a*) (3, 4) (*b*) (4, 5) (*c*) (4, 3) (*d*) (5, 4)

6. The roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ are
   (*a*) $\frac{c-a}{b-c}, 1$ (*b*) $\frac{a-b}{b-c}, 1$
   (*c*) $\frac{b-c}{a-b}, 1$ (*d*) $\frac{c-a}{a-b}, 1$

7. If a, b, c are integers and $b^2 = 4(ac + 5d^2)$, $d \in N$, then roots of the equation $ax^2 + bx + c = 0$ are
   (*a*) Irrational (*b*) Rational & different
   (*c*) Complex conjugate (*d*) Rational & equal

8. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [5, 100]$. Total number of different values of 'n' so that the given equation has integral roots, is
   (*a*) 4 (*b*) 6 (*c*) 8 (*d*) 3

9. If $a, b \in R, a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then $a + b + 1$ is:
   (*a*) Positive (*b*) Negative
   (*c*) Zero (*d*) Depends on the sign of b

10. If a and b are the non-zero distinct roots of $x^2 + ax + b = 0$, then the least value of $x^2 + ax + b$ is
    (*a*) $\frac{3}{2}$ (*b*) $\frac{9}{4}$ (*c*) $-\frac{9}{4}$ (*d*) 1

11. If both roots of the quadratic equation $(2 - x)(x + 1) = p$ are distinct & positive, then p must lie in the interval
    (*a*) $(2, \infty)$ (*b*) $(2, 9/4)$
    (*c*) $(-\infty, -2)$ (*d*) $(-\infty, \infty)$

12. If $(1 - p)$ is root of quadratic equation $x^2 + px + (1 - p) = 0$, then its roots are
    (*a*) 0, 1 (*b*) −1, 1 (*c*) 0, −1 (*d*) −1, 2

13. If $a + b + c = 0$ then the equation $3ax^2 + 2bx + c = 0$ has
    (*a*) Imaginary roots
    (*b*) Real and equal roots
    (*c*) Real and different roots
    (*d*) Rational roots

14. If a, b, c are in A.P then the roots of the equation $ax^2 + 2bx + c = 0$ are $(a \neq b \neq c)$
    (*a*) Real and distinct (*b*) Real and equal
    (*c*) Real (*d*) Imaginary

15. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is
    (*a*) Greater than $4ab$ (*b*) Less than $4ab$
    (*c*) Greater than $-4ab$ (*d*) Less than $-4ab$

SUM AND PRODUCT OF ROOTS

16. If one root of $5x^2 + 13x + k = 0$ is reciprocal of the other, then $k =$
    (*a*) 0 (*b*) 5 (*c*) 1/6 (*d*) 6

Credit:--- Divansh.....

17. If α and β are the roots of the equation $4x^2 + 3x + 7 = 0$, then $\frac{1}{\alpha} + \frac{1}{\beta} =$

(*a*) $-\frac{3}{7}$ (*b*) $\frac{3}{7}$ (*c*) $-\frac{3}{5}$ (*d*) $\frac{3}{5}$

18. If α, β be the roots of the equation $x^2 - 2x + 3 = 0$, then the equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ is

(*a*) $x^2 + 2x + 1 = 0$ (*b*) $9x^2 + 2x + 1 = 0$
(*c*) $9x^2 - 2x + 1 = 0$ (*d*) $9x^2 + 2x - 1 = 0$

19. If the product of roots of the equation, $mx^2 + 6x + (2m - 1) = 0$ is -1, then the value of m will be

(*a*) 1 (*b*) -1 (*c*) $\frac{1}{3}$ (*d*) $-\frac{1}{3}$

20. If α, β, γ are the roots of the equation $x^3 + x + 1 = 0$, then the value of $\alpha^3 \beta^3 \gamma^3$

(*a*) 0 (*b*) -3 (*c*) 3 (*d*) -1

21. If a, b are the roots of quadratic equation $x^2 + px + q = 0$ and g, d are the roots of $x^2 + px - r = 0$, then $(a - g).(a - d)$ is equal to:

(*a*) $q + r$ (*b*) $q - r$
(*c*) $-(q + r)$ (*d*) $-(p + q + r)$

22. If α, β are roots of the equation $ax^2 + bx + c = 0$, then the value of $\alpha^3 + \beta^3$ is

(*a*) $\frac{3abc + b^3}{a}$ (*b*) $\frac{a^3 + b^3}{3abc}$

(*c*) $\frac{3abc - b^3}{a^3}$ (*d*) $\frac{-(3abc + b^3)}{a^3}$

3. If $\alpha + \beta = -2$ and $\alpha^3 + \beta^3 = -56$ then the quadratic equation whose roots are α, β is

(*a*) $x^2 + 2x - 16 = 0$ (*b*) $x^2 + 2x - 15 = 0$
(*c*) $x^2 + 2x - 12 = 0$ (*d*) $x^2 + 2x - 8 = 0$

24. tan 22° and tan 23° are roots of $x^2 + ax + b = 0$ then

(*a*) $a + b + 1 = 0$ (*b*) $a - b + 1 = 0$
(*c*) $b - a + 1 = 0$ (*d*) $a + b = 1$

25. Let α, β be the roots of the equation $(x - a)(x - b) = c$, $c \neq 0$ then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are

(*a*) a, c (*b*) b, c
(*c*) a, b (*d*) $a + b, b + c$

26. Let α, β are the roots of the equation $x^2 - 15x + 1 = 0$, then the value of $\left(\frac{1}{\alpha} - 15\right)^{-2} + \left(\frac{1}{\beta} - 15\right)^{-2}$ is

(*a*) 225 (*b*) 900 (*c*) 223 (*d*) 0

27. If one root of $x^2 + px + 12 = 0$ is 4 while the equation $x^2 + px + q = 0$ has equal roots then $q =$

(*a*) -7 (*b*) 4 (*c*) 42 (*d*) $\frac{49}{4}$

28. If one root of the equation $ax^2 + bx + c = 0$ is the square of the other, then

(*a*) $b^2 + ac^2 + a^2c = 3abc$
(*b*) $b^3 + ac^2 + a^2c = 3abc$
(*c*) $b^2 + ac^2 + a^2c + 3abc = 0$
(*d*) $b^3 + ac^2 + a^2c + 3abc = 0$

29. The number of real solution of the equation $(9/10)^x = -3 + x - x^2$ is

(*a*) 2 (*b*) 0 (*c*) 1 (*d*) 3

30. If the roots of $x^2 - bx + c = 0$ are two consecutive integers then $b^2 - 4c =$

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

31. If the roots of the quadratic equation $x^2 + px + q$ are tan α and tan β respectively then the value of $\frac{p}{q - 1}$ is

(*a*) $\tan(\alpha - \beta)$ (*b*) $\cot(\alpha - \beta)$
(*c*) $\tan(\alpha + \beta)$ (*d*) $\cot(\alpha + \beta)$

COMMON ROOTS

32. If the equation $x^2 + px + q = 0$ and $x^2 + qx + p = 0$, have a common root, (Where $p \neq q$) then $p + q + 1 =$

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) -1

33. If a, b, p, q are non-zero real numbers, then the equations $2a^2x^2 - 2\,abx + b^2 = 0$ and $p^2x^2 + 2pqx + q^2 = 0$ have:

(*a*) No common root
(*b*) One common root if $2a^2 + b^2 = p^2 + q^2$
(*c*) Two common roots if $3pq = 2ab$
(*d*) Two common roots if $3qb = 2ap$

34. The equations $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have a common root. If $a, b, c \in N$ then the least possible value of $a + b + c$ is equal to

(*a*) 3 (*b*) 6 (*c*) 9 (*d*) 12

THEORY OF EQUATION AND IDENTITY, INEQUALITIES

35. The set of all real numbers x for which $x^2 - |x + 2| + x > 0$ is

(*a*) $(-\infty, -2) \cup (2, \infty)$ (*b*) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(*c*) $(-\infty, -1) \cup (1, \infty)$ (*d*) $(\sqrt{2}, \infty)$

36. Number of values of 'p' for which the equation $(p^2 - 3p + 2)x^2 - (p^2 - 5p + 4)x + p - p^2 = 0$ possess more than two roots, is:

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) None of these

37. The complete solution set of the inequality $\frac{x^4 - 3x^3 + 2x^2}{x^2 - x - 30} \geq 0$ is:

(*a*) $(-\infty, -5) \cup (1, 2) \cup (6, \infty) \cup \{0\}$
(*b*) $(-\infty, -5) \cup [1, 2] \cup (6, \infty) \cup \{0\}$
(*c*) $(-\infty, -5] \cup [1, 2] \cup [6, \infty) \cup \{0\}$
(*d*) None of these

38. If the inequality $(m-2)x^2 + 8x + m + 4 > 0$ is satisfied for all $x \in R$, then the least integral 'm' is:

(a) 4 (b) 5
(c) 6 (d) None of these

39. The set of all solutions of the inequality $(1/2)^{x^2-2x} < 1/4$ contains the set

(a) $(-\infty, 0)$ (b) $(-\infty, 1)$ (c) $(1, \infty)$ (d) $(3, \infty)$

40. If two roots of the equation $x^3 - px^2 + qx - r = 0$ are equal in magnitude but opposite in sign, then:

(a) $pr = q$ (b) $qr = p$
(c) $pq = r$ (d) None of these

41. If α, β & γ are the roots of the equation $x^3 - x - 1 = 0$ then. $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$ has the value equal to:

(a) Zero (b) -1 (c) -7 (d) 1

LOCATION OF ROOTS

42. The real values of 'a' for which the quadratic equation $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a = 0$ possesses roots of opposite sign is given by:

(a) $a > 5$ (b) $0 < a < 4$
(c) $a > 0$ (d) $a > 7$

43. If a, b are the roots of the quadratic equation $x^2 - 2p(x-4) - 15 = 0$, then the set of values of 'p' for which one root is less than 1 and the other root is greater than 2 is:

(a) $(7/3, \infty)$ (b) $(-\infty, 7/3)$
(c) $x \in R$ (d) None of these

44. The values of k, for which the equation $x^2 + 2(k-1)x + k + 5 = 0$ possesses atleast one positive root, are

(a) $[4, \infty)$ (b) $(-\infty, -1] \cup [4, \infty)$
(c) $[-1, 4]$ (d) $(-\infty, -1]$

45. The set of values of p for which 6 lies between the roots of the equation $x^2 + 2(p-3)x + 9 = 0$ is

(a) $\left(\frac{3}{4}, \infty\right)$ (b) $\left(-\infty, -\frac{3}{4}\right)$

(c) $\left(-\frac{3}{4}, \frac{3}{4}\right)$ (d) $\left[-\frac{3}{4}, \frac{3}{4}\right]$

46. If $\left(\frac{x-1}{x+2}\right)^3 = \frac{x-4}{x+5}$, then $x =$

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) $\frac{1}{4}$

47. If α, β, γ are the roots of the equation $x^3 - 4x + 1 = 0$, then find the value of $\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha} =$

(a) 1 (b) 2 (c) 3 (d) 4

48. The solution of equation $x^3 - 13x^2 + 15x + 189 = 0$, if one root exceeds the other by 2 is given by

(a) $7, 9, -3$ (b) $2, -5, -4$ (c) $1, 2, 6$ (d) $-1, -3, 9$

49. If the inequality $\frac{mx^2 + 3x + 4}{x^2 + 2x + 2} < 5$ is satisfied for all $x \in R$, then the values of m

(a) $1 < m < 5$ (b) $-1 < m < 5$
(c) $1 < m < 6$ (d) $m < 71/24$

50. Sum of all the roots of the equation $x^2 - 2x - |x - 1| - 5 = 0$ is

(a) 0 (b) 2 (c) 1 (d) 5

Prabal (JEE Main Level)

1. The value of k for which the equation $(k-2)x^2 + 8x + k + 4 = 0$ has both real, distinct and negative roots is

(a) 0 (b) 2 (c) 3 (d) -4

2. If α, β are the roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are and $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$, is

(a) $acx^2 + (a+c)bx + (a+c)^2 = 0$
(b) $abx^2 + (a+c)bx + (a+c)^2 = 0$
(c) $acx^2 + (a+b)cx + (a+c)^2 = 0$
(d) $acx^3 + (a+c)bx + (a+c)^3$

3. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio $2:3$, then $m =$

(a) $5\sqrt{10}$ (b) $3\sqrt{10}$ (c) $2\sqrt{10}$ (d) $10\sqrt{5}$

4. $x^2 - 11x + a$ and $x^2 - 14x + 2a$ will have a common factor, if $a =$

(a) 24 (b) 0, 24
(c) 3, 24 (d) 0, 3

5. The value of p for which both the roots of the equation $4x^2 - 20px + (25p^2 + 15p - 66) = 0$ are less than 2, lies in

(a) $(4/5, 2)$ (b) $(2, \infty)$
(c) $(-1, -4/5)$ (d) $(-\infty, -1)$

6. The number of quadratic equations which are unchanged by squaring their roots

(a) 2 (b) 3
(c) 4 (d) None of these

Credit:--- Divansh.....

7. If $b^2 < 2ac$, then equation $ax^3 + bx^2 + cx + d = 0$ has

(*a*) Exactly one real roots
(*b*) Has three real roots
(*c*) At least two roots
(*d*) None of these

8. If $\sec\alpha$, $\tan\alpha$ are roots of $ax^2 + bx + c = 0$, then

(*a*) $a^4 - b^4 + 4ab^2c = 0$ (*b*) $a^4 + b^4 - 4ab^2c = 0$
(*c*) $a^2 - b^2 = 4ac$ (*d*) $a^2 + b^2 = ac$

9. If one root of the equation $(l - m)x^2 + lx + 1 = 0$ is double the other and if l is real, then the greatest value of m is

(*a*) $\frac{9}{8}$ (*b*) $\frac{7}{8}$ (*c*) $\frac{8}{9}$ (*d*) $\frac{8}{9}$

10. If p, q, r are real numbers satisfying the condition $p + q + r = 0$, then the roots of the quadratic equation $3px^2 + 5qx + 7r = 0$ are

(*a*) Positive (*b*) Negative
(*c*) Real and distinct (*d*) Imaginary

11. If a, b, c are in G.P., then the equation $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

(*a*) A.P. (*b*) G.P.
(*c*) H.P. (*d*) None of these

12. The number of integral roots of the equation $x^8 - 24x^7 - 18x^5 + 39x^2 + 1155 = 0$ is:

(*a*) 0 (*b*) 2 (*c*) 4 (*d*) 6

13. If $\alpha \neq \beta$ but $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$, then the equation with roots $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ is:

(*a*) $3x^2 - 25x + 3 = 0$ (*b*) $x^2 + 5x - 3 = 0$
(*c*) $x^2 - 5x + 3 = 0$ (*d*) $3x^2 - 19x + 3 = 0$

14. Minimum possible number of positive root of the quadratic equation $x^2 - (1 + \lambda)x + \lambda - 2 = 0$, $\lambda \in R$:

(*a*) 2 (*b*) 0
(*c*) 1 (*d*) Can not be determined

15. If $\alpha, \beta, \gamma, \delta \in R$ satisfy

$$\frac{(\alpha+1)^2+(\beta+1)^2+(\gamma+1)^2+(\delta+1)^2}{\alpha+\beta+\gamma+\delta} = 4$$

If biquadratic equation $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ has the roots $\left(\alpha+\frac{1}{\beta}-1\right), \left(\beta+\frac{1}{\gamma}-1\right), \left(\gamma+\frac{1}{\delta}-1\right), \left(\delta+\frac{1}{\alpha}-1\right)$. Then the value of a_2/a_0 is:

(*a*) 4 (*b*) –4
(*c*) 6 (*d*) None of these

16. Both the roots of the equation $(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$ are always

(*a*) positive (*b*) real
(*c*) negative (*d*) None of these.

17. If $(x^2 + px + 1)$ is a factor of $(ax^3 + bx + c)$, then

(*a*) $a^2 + c^2 = -ab$ (*b*) $a^2 - c^2 = -ab$
(*c*) $a^2 - c^2 = ab$ (*d*) None of these

18. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

(*a*) 4 (*b*) 1 (*c*) 3 (*d*) 2

19. Two towns A and B are 60km apart. A school is to be built to serve 150 students in town A and 50 students in town B. If the total distance to be travelled by all 200 student is to be as small as possible, then the school should be built at

(*a*) town B (*b*) 45 km from town A
(*c*) town A (*d*) 45 km from town B

20. If p, q, r are +ve and are in $A.P.$, the roots of quadratic equation $px^2 + qx + r = 0$ are all real for

(*a*) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (*b*) $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$
(*c*) all p and r (*d*) no p and r

21. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots is

(*a*) 15 (*b*) 9 (*c*) 7 (*d*) 8

22. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

(*a*) $0 < \alpha < \beta$ (*b*) $\alpha < 0 < \beta < |\alpha|$
(*c*) $\alpha < \beta < 0$ (*d*) $\alpha < 0 < |\alpha| < \beta$

23. If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, then $M = (a + b)(c + d)$ satisfies the relation

(*a*) $0 \leq M \leq 1$ (*b*) $1 \leq M \leq 2$
(*c*) $2 \leq M \leq 3$ (*d*) $3 \leq M \leq 4$

24. For the equation $3x^2 + px + 3 = 0, p > 0$, if one of the roots is square of the other, then p is equal to

(*a*) 1/3 (*b*) 1 (*c*) 3 (*d*) 2/3

25. If $a_1, a_2 \ldots a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 2a_n$ is

(*a*) $n(2c)^{1/n}$ (*b*) $(n+1)c^{1/n}$
(*c*) $2nc^{1/n}$ (*d*) $(n+1)(2c)^{1/n}$

26. Number of positive integers n for which $n^2 + 96$ is a perfect square is

(*a*) 4 (*b*) 8 (*c*) 12 (*d*) infinite

27. If $x^2 + 5 = 2x - 4\cos(a + bx)$, where $a, b \in (0, 5)$, is satisfied for at least one real x, then the maximum value of $a + b$ is equal to

(*a*) 3π (*b*) 2π (*c*) π (*d*) $-\pi$

28. Let r, s, t be the roots of the equation $x^3 + ax^2 + bx + c = 0$, such that $(rs)^2 + (st)^2 + (rt)^2 = b^2 - kac$, then $k =$

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

29. Let a, b, c, d be distinct real numbers such that a, b are roots of $x^2 - 5cx - 6d = 0$, and c, d are roots of $x^2 - 5ax - 6b = 0$. Then $b + d$ is

(*a*) 180 (*b*) 162 (*c*) 144 (*d*) 126

30. Let $S = \{1, 2, 3, \ldots, 100\}$. Suppose b and c are chosen at random from the set S. The probability that $4x^2 + bx + c$ has equal roots is

(*a*) 0.001 (*b*) 0.004 (*c*) 0.007 (*d*) 0.01

Credit:--- Divansh.....

31. Let a_1, a_2, a_3, a_4 be real numbers such that $a_1 + a_2 + a_3 + a_4 = 0$ and $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. Then the smallest possible value of the expression $(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4)^2 + (a_4 - a_1)^2$ lies in the interval

(*a*) (0, 1.5) (*b*) (1.5, 2.5) (*c*) (2.5, 3) (*d*) (3, 3.5)

32. Let $r(x)$ be the remainder when the polynomial $x^{135} + x^{125} - x^{115} + x^5 + 1$ is divided by $x^3 - x$. Then

(*a*) $r(x)$ is the zero polynomial

(*b*) $r(x)$ is a non-zero constant

(*c*) degree of $r(x)$ is one

(*d*) degree or $r(x)$ is two

33. The sum of all non-integer roots of the equation $x^5 - 6x^4 + 11x^3 - 5x^2 - 3x + 2 = 0$ is

(*a*) 6 (*b*) −11 (*c*) −5 (*d*) 3

34. The least value of the expression $x^2 + 4y^2 + 3z^2 - 2x - 12y - 6z + 14$ is

(*a*) 1 (*b*) no least value

(*c*) 0 (*d*) None of these

35. If the equations $px^2 + qx + r = 0$ and $rx^2 + qx + p = 0$ $(p \neq r \neq 0)$ have a negative common root then the value of $(p - q + r)$ is equal to

(*a*) −1 (*b*) 1 (*c*) 2 (*d*) 0

36. Let α and β two real roots of the equation $(k + 1)\tan^2 x - \sqrt{2}\,.\,\lambda \tan x = (1 - k)$, where $k\ (\neq -1)$ and λ are real numbers. If $\tan^2(\alpha + \beta) = 50$, then a value of λ is

(*a*) 5 (*b*) 10 (*c*) $10\sqrt{2}$ (*d*) $5\sqrt{2}$

37. Let α and β be the roots of the equation, $5x^2 + 6x - 2 = 0$. If $S_n = \alpha^n + \beta^n$, $n = 1, 2, 3, \ldots$, then

(*a*) $5S_6 + 6S_5 + 2S_4 = 0$ (*b*) $6S_6 + 5S_5 = 2S_4$

(*c*) $6S_6 + 5S_5 + 2S_4 = 0$ (*d*) $5S_6 + 6S_5 = 2S_4$

38. Let p and q be two positive numbers such that $p + q = 2$ and $p^4 + q^4 = 272$. Then p and q are roots of the equation

(*a*) $x^2 - 2x + 2 = 0$ (*b*) $x^2 - 2x + 8 = 0$

(*c*) $x^2 - 2x + 136 = 0$ (*d*) $x^2 - 2x + 16 = 0$

39. The number of all 3-digit integers abc for which $(a \times b \times c) + (a \times b)6 + (c \times a) + a + b + c = 29$ is

(*a*) 6 (*b*) 10 (*c*) 14 (*d*) 18

NUMERICAL TYPE QUESTION

40. If solutions set of $\sqrt[3]{6(5x+6)} - \sqrt[3]{5(6x-11)} = 1$. is $[a, -b/5a]$ then $a + b =$

41. Let p and q be the two roots of the equation $mx^2 + x(2 - m) + 3 = 0$. If m_1, m_2 are the two values of m satisfying $\frac{p}{q} + \frac{q}{p} = \frac{2}{3}$ the value of $\frac{m_1}{m_2^2} + \frac{m_2}{m_1^2}$ is

42. The minimum number of real roots of $(x^2 + 3x + a)(x^2 + ax + 1) = 0$ equals

43. The number of order pairs of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$ is

44. Given that α, β, γ are the roots of cubic equation

$$x^3 - 3x^2 + 2x + \frac{2}{3} = 0.$$

The value of $\alpha^4 + \beta^4 + \gamma^4$ is equal to ______

45. Let $P(x)$ be a polynomial whose coefficient are integer satisfies $P(1) = 5$ and $P(2) = 7$. The smallest possible positive value of $P(10)$ is equal to ________

46. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $p_n = (\alpha)^n + (\beta)^n$, $p_{n-1} = 11$ and $p_{n+1} = 29$ for some integer $n \geq 1$. Then, the value of is p_n^2 is ____.

47. Let the quadratic equation $x^2 + 3x - k = 0$ has roots a, b and $x^2 + 3x - 10 = 0$ has roots c, d such that modulus of difference of the roots of the first equation is equal to twice the modulus of the difference of the roots of the second equation. If the value of 'k' can be expressed as rational number in the lowest form as m/n then find the value of $(m + n)$.

48. When $y^2 + my + 2$ is divided by $(y - 1)$ then the quotient is $f(y)$ and the remainder is R_1. When $y^2 + my + 2$ is divided by $(y + 1)$ then quotient is $g(y)$ and the remainder is R_2. If $R_1 = R_2$ then find the value of m.

49. Let $f(x)$ be the remainder when $p(x) = x^{73} - 2x^6 + 3x^3$ is divided by $x^4(x^2 - 1)$ then $f(1) =$

50. If all the values of 'a', so that exactly one root of the equation $x^2 - 2ax + a^2 - 1 = 0$, lies between the numbers 2 and 4, and no root of the equation is either equal to 2 or equal to 4 is $(0, \lambda) \cup (\lambda, 5)$ then find λ.

Parikshit (JEE Advanced Level)

SINGLE CORRECT TYPE QUESTIONS

1. Let a, b, c be the sides of a triangle where $a \neq b \neq c$ and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then

(*a*) $\lambda < \frac{4}{3}$ (*b*) $\lambda < \frac{5}{3}$

(*c*) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (*d*) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

2. The number of real roots of equation $x^8 - x^5 + x^2 - x + 2 = 0$ is

(*a*) 2 (*b*) 4 (*c*) 6 (*d*) 0

3. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$.

Which of the following interval(s) is a subset of S?

(a) $\left(-\frac{1}{2},-\frac{1}{3}\right)$ (b) $\left(\frac{1}{\sqrt{10}},\frac{1}{2}\right)$

(c) $\left(\frac{1}{\sqrt{5}},\frac{1}{2}\right)$ (d) None of these

4. Polynomials $f(x)$ and $g(x)$ are defined as follows:
$f(x) = x^2 + ax + 2; g(x) = x^2 + 2x + a. (a \neq 2)$
If the equation $f(x) = 0$ and $g(x) = 0$ have a common root then the sum of the roots of the equation $f(x) + g(x) = 0$ is.

(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{2}$ (d) 1

5. If $a, b, c \in R, a \neq 0$ and $(b - 1)^2 < 4ac$; then the number of real roots of the system of equation (in three unknowns x_1, x_2, x_3)

$ax_1^2 + bx_1 + c = x_2,\ \ ax_2^2 + bx_2 + c = x_3,\ ax_3^2 + bx_3 + c = x_1$ is

(a) 0 (b) 1 (c) 2 (d) 3

6. The values of a and b so that $x^4 + 12x^3 + 46x^2 + ax + b$ is square of quadratic expression are respectively

(a) 60, 25 (b) 45, 25 (c) 60, 30 (d) 25, 45

7. Values of k for which the inequality $k\sin^2 x - k\sin x + 1 \geq 0$ is true $\forall x \in R$ is

(a) $k > -\frac{1}{2}$ (b) $k > 4$

(c) $-\frac{1}{2} \leq k \leq 4$ (d) $\frac{1}{2} \leq k \leq 5$

8. Consider the following statements:
S_1 : If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then value of $b^2 - 4c$ is equal to 1.
S_2 : if α, β are roots of $x^2 - x + 3 = 0$ then value of $\alpha^4 + \beta^4$ is equal 7.
S_3 : If α, β γ are the roots of $x^3 - 7x^2 + 16x - 12 = 0$ then value of $\alpha^2 + \beta^2 + \gamma^2$ is equal to 17.
State, in order, whether S_1, S_2, S_3 are true or false

(a) TTT (b) FTF (c) TFT (d) FTT

9. The complete set of values of 'b' for which the equation $2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2)$ has only one solution.

(a) $(-\infty,-14)\cup\left[\frac{14}{3},\infty\right)\cup\{4\}$

(b) $(-\infty,-14)\cup\{4\}\cup\left[\frac{14}{3},\infty\right)\cup\{-12\}$

(c) $(-\infty,-14)\cup\left[\frac{14}{3},\infty\right)\cup\{-12\}$

(d) $(-\infty,-14)\cup\left[\frac{14}{3},\infty\right)$

10. Let p and q be real numbers such that $p = 0, p^3 = q$ and $p^3 = -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$; then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is

(a) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(b) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(c) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(d) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

11. The set of values of $K(K \in R)$ for which the equation $x^2 - 4|x| + 3 - |K^2 - 1| = 0$ will have exactly four real roots is

(a) $(-2, 4)$ (b) $(-4, 4)$

(c) $(-2, 2)$ (d) $(-1, 0)$

12. If $\left|x^2 - \frac{\pi}{4} + \sin x - 2\right| = |x^2 - 2| + |\sin x| + \frac{\pi}{4}$ then

(a) $x \in (0, \sqrt{2})$ (b) $x \in (-\sqrt{2}, \sqrt{2})$

(c) $x \in R$ (d) $x \in (-\sqrt{2}, 0)$

13. Let $\alpha = \max\limits_{x\in R}\{8^{2\sin 3x} \cdot 4^{4\cos 3x}\}$ and $\min\limits_{x\in R}\{8^{2\sin 3x} \cdot 4^{4\cos 3x}\}$. If $8x^2 + bx + c = 0$ is a quadratic equation whose roots are $\alpha^{1/5}$ and $\beta^{1/5}$, then the value of $c - b$ is equal to

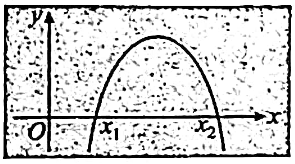
(a) 42 (b) 43 (c) 47 (d) 50

MULTIPLE CORRECT TYPE QUESTIONS

14. If the equation $ax^2 + bx + c = 0$ $(a > 0)$ has two real roots α and β such that $\alpha < -2$ and $\beta > 2$, then which of the following statements is/are true?

(a) $a - |b| + c < 0$
(b) $c < 0, b^2 - 4ac > 0$
(c) $4a - 2|b| + c < 0$
(d) $9a - 3|b| + c < 0$

15. The adjoining figure shows the graph of $y = ax^2 + bx + c$. Then



(a) $a < 0$
(b) $b^2 < 4ac$
(c) $c > 0$
(d) a and b are of opposite sign

16. If $(x^2 + x + 1) + (x^2 + 2x + 3) + (x^2 + 3x + 5) + ... + (x^2 + 20x + 39) = 4500$, then x is equal to

(a) 10 (b) -10 (c) 20.5 (d) -20.5

17. If roots of equation, $x^3 + bx^2 + cx - 1 = 0$ forms an increasing G.P., then

(a) $b + c = 0$
(b) $b \in (-\infty, -3)$
(c) one of the roots = 1
(d) one root is smaller than 1 & other > 1

Credit:--- Divansh.....

18. If $x \in R$, then the roots of the equation $x^4 + 4x^3 - 8x^2 + k = 0$ with respect to the values of k, ($k \in [0, 2014]$), are

(*a*) for $k \in [0, 3]$: 4 real and distinct roots
(*b*) for $k \in (3, 128)$: two real and two imaginary roots
(*c*) for $k \in (128, 583)$: no real root
(*d*) for $k \in (0, 3)$: four real and distinct roots

19. If a, b are non-zero real numbers and α, β the roots of $x^2 + ax + b = 0$, then

(*a*) α^2, β^2 are the roots of $x^2 - (2b - a^2)x + a^2 = 0$
(*b*) $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots of $bx^2 + ax + 1 = 0$
(*c*) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of $bx^2 + (2b - a^2)x + b = 0$
(*d*) $(\alpha - 1)$, $(\beta - 1)$ are the roots of the equation $x^2 + x(a+2) + 1 + a + b = 0$

20. Let $f(x) = \frac{3}{x-2} + \frac{4}{x-3} + \frac{5}{x-4}$, then $f(x) = 0$ has

(*a*) exactly one real root in (2, 3)
(*b*) exactly one real root in (3, 4)
(*c*) at least one real root in (2, 3)
(*d*) None of these

COMPREHENSION BASED QUESTIONS

Comprehension (Q. No. 21 to 23):

Consider an unknown polynomial which when divided by $(x - 3)$ and $(x - 4)$ leaves remainders 2 and 1, respectively. Let $R(x)$ be the remainder when this polynomial is divided by $(x - 3)(x - 4)$

21. If equation $R(x) = x^2 + ax + 1$ has two distinct real roots, then exhaustive values of a are

(*a*) $(-2, 2)$ (*b*) $(-\infty, -2) \cup (2, \infty)$
(*c*) $(-2, \infty)$ (*d*) All real numbers

22. If $R(x) = px^2 + (q - 1)x + 6$ has no distinct real roots and $p > 0$, then the least value of $3p + q$ is

(*a*) –2 (*b*) 2/3
(*c*) –1/3 (*d*) None of these

23. Range of $f(x) = [R(x)] / (x^2 - 3x + 2)$ is

(*a*) $[-2, 2]$
(*b*) $(-\infty, -2-\sqrt{3}] \cup [-2+\sqrt{3}, \infty)$
(*c*) $(-\infty, -7-4\sqrt{3}] \cup [-7+4\sqrt{3}, \infty)$
(*d*) None of these

Comprehension (Q. No. 24 to 26):

Consider the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$, where $a \in R$.

Also, range of $f(x) = x + \frac{1}{x}$ is $(-\infty, -2] \cup [2, \infty)$

24. If equation has at least two distinct positive real roots, then all possible value of 'a' are

(*a*) $\left(-\infty, -\frac{1}{4}\right)$ (*b*) $\left(\frac{5}{4}, \infty\right)$ (*c*) $\left(-\infty, -\frac{3}{4}\right)$ (*d*) $\left(\frac{-5}{4}, \infty\right)$

25. If equation has at least two distinct negative real roots, then all possible values of 'a' are

(*a*) $\left(\frac{3}{4}, \infty\right)$ (*b*) $\left(-\frac{5}{4}, \infty\right)$
(*c*) $\left(-\infty, \frac{1}{4}\right)$ (*d*) All of the above

26. If exactly two roots are positive and two roots are negative, then number of integral values of 'a' is

(*a*) 2 (*b*) 1 (*c*) 0 (*d*) 3

Comprehension (Q. No. 27 to 29):

Consider the equation $x^4 - \lambda x^2 + 9 = 0$. This can be solved by substituting $x^2 = t$ such equations are called as pseudo quadratic equations.

27. If the equation has four real and distinct roots, then λ lies in the interval

(*a*) $(-\infty, -6) \cup (6, \infty)$ (*b*) $(0, \infty)$
(*c*) $(6, \infty)$ (*d*) $(-\infty, -6)$

28. If the equation has no real root, then λ lies in the interval

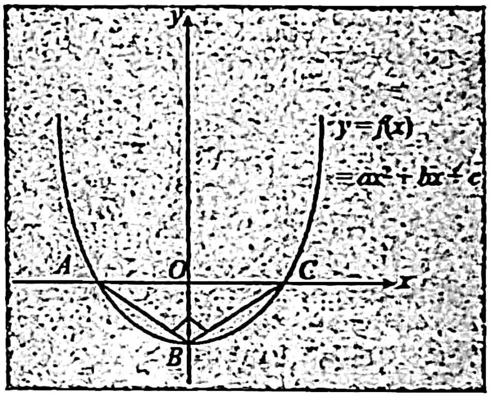
(*a*) $(-\infty, 0)$ (*b*) $(-\infty, 6)$
(*c*) $(6, \infty)$ (*d*) $(0, \infty)$

29. If the equation has only two real roots, then set of values of λ is

(*a*) $(-\infty, -6)$ (*b*) $(-6, 6)$ (*c*) $\{6\}$ (*d*) ϕ

Comprehension (Q. No. 30 to 32)

If vertices of ΔABC lie on $y = f(x) = ax^2 + bx + c$. The ΔABC is right-angled isosceles triangle whose hypotenuse $AC = 4\sqrt{2}$ units.



30. $y = f(x)$ is given by

(*a*) $y = x^2 - 2\sqrt{2}$ (*b*) $y = x^2 - 12$
(*c*) $y = \frac{x^2}{2} - 2$ (*d*) $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

31. Minimum value of $y = f(x)$ is

(*a*) –4 (*b*) –2
(*c*) $-2\sqrt{2}$ (*d*) None of these

32. Number of integral values of k for which one roots of $f(x) = 0$ is more than k and other less than k

(*a*) 6 (*b*) 5 (*c*) 5 (*d*) 7

Credit:--- Divansh.....

Comprehension (Q. No. 33 to 35) Consider the inequality $9^x - a3^x - a + 3 \leq 0$, where a is real parameter.

33. The given inequality has atleast one negative solution for $a \in$

    (*a*) $(-\infty, 2)$ (*b*) $(3, \infty)$ (*c*) $(-2, \infty)$ (*d*) $(2, 3)$

34. The given inequality has atleast one positive solution for $a \in$

    (*a*) $(-\infty, 2)$ (*b*) $[3, \infty)$ (*c*) $(2, \infty)$ (*d*) $[-2, \infty)$

35. The given inequality has atleast one real solution for $a \in$

    (*a*) $(-\infty, 3)$ (*b*) $[2, \infty)$ (*c*) $(3, \infty)$ (*d*) $[-2, \infty)$

Comprehension (Q. No. 36 to 38): Let $f(x) = ax^2 + bx + c$; $a, b, c \in R$

It is given $|f(x)| \leq 1$, $\forall\ |x| \leq 1$.

Now, answer the following questions.

36. The possible value of $|a + c|$, if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by

    (*a*) 1 (*b*) 0 (*c*) 2 (*d*) 3

37. The possible value of $|a + b|$, if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by

    (*a*) 1 (*b*) 0 (*c*) 2 (*d*) 3

38. The possible maximum value of $\frac{8}{3}a^2 + 2b^2$ is given by

    (*a*) 32 (*b*) $\frac{32}{3}$ (*c*) $\frac{2}{3}$ (*d*) $\frac{16}{3}$

MATCH THE COLUMN TYPE QUESTIONS

39. Match the column.

	Column-I		Column-II
A.	If α, $\alpha + 4$ are two roots of $x^2 - 8x + k = 0$, then possible value of k is	p.	2
B.	Number of real roots of equation $x^2 - 5\|x\| + 6 = 0$ are 'n', then value of $n/2$ is	q.	3
C.	If $3 - i$ is a root of $x^2 + ax + b = 0$ $(a, b \in R)$, then b is	r.	12
D.	If both roots of $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then 'k' may be equal to	s.	10

(*a*) A → r, B → p, C → s, D → p,q

(*b*) A → r, B → p, C → s, D → p,r

(*c*) A → r, B → q, C → s, D → p

(*d*) A → p, B → r, C → s, D → q

40. Match the following for the equation $x^2 + a|x| + 1 = 0$, where a is a parameter.

	Column-I		Column-II
A.	No real roots for	p.	$a < -2$
B.	Two real roots for	q.	$a \in \phi$
C.	Three real roots for	r.	$a = -2$
D.	Four distinct real roots for	s.	$a \geq 0$

(*a*) A → p, B → r, C → q, D → s

(*b*) A → s, B → r, C → q, D → p

(*c*) A → s, B → q, C → r, D → p

(*d*) A → p, B → q, C → r, D → s

41. Match the column.

	Column-I		Column-II
A.	If a, b, c, and d are four non-zero real number such that $(d + a - b)^2 + (d + b - c)^2 = 0$ and the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are real and equal. Then	p.	$a + b + c = 0$
B.	If the roots of the equation $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$ are real and equal, then	q.	a,b,c are in *A.P*
C.	If the equation $ax^2 + bx + c = 0$ and $x^3 - 3x^2 + 3x - 1 = 0$ have a common real root, then	r.	a,b,c are in *G.P*
D.	Let a, b, c be positive real number such that the expression $bx^2 + (\sqrt{(a+c)^2 + 4b^2})\ x + (a + c)$ is non-negative $\forall\ x \in R$, then	s.	a,b,c are in *H.P.*

(*a*) A → p, B → r, C → p, D → q

(*b*) A → q,s, B → p, C → r, D → p

(*c*) A → q,s, B → r, C → p, D → q

(*d*) A → p, B → r, C → p, D → q,s

42. Match the column.

	Column-I		Column-II
A.	One roots is positive and the other is negative for the equation $(m-2)x^2 - (8-2m)x - (8-3m) = 0$, then number of Integral values of m	p.	0
B.	Exactly one root of equation $x^2 - m(2x - 8) - 15 = 0$ lies in interval $(0, 1)$, then number of Integral values of m	q.	infinity solution
C.	The equation $x^2 + 2(m + 1)x + 9m - 5 = 0$ has both roots negative, then number of Integral values of m	r.	1

D.	The equation $x^2 + 2(m - 1)x + m + 5 = 0$ has both roots lying on either sides of 1, then number of Integral values of m	s.	2

(*a*) A → q, B → r, C → q, D → p

(*b*) A → p, B → r, C → q, D → q

(*c*) A → p, B → r, C → q, D → s

(*d*) A → p, B → q, C → r, D → q

INTEGER TYPE QUESTIONS

43. Find number of integral roots of equation $x(x+1)(x+2)(x+3) = 120$.

44. Find product of all real values of x satisfying
$$(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3} = 10$$

45. The least prime integral value of '$2a$' such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality $\frac{\alpha}{\beta}+\frac{\beta}{\alpha} < 2$ is

46. If a, b are the roots of $x^2 + px + 1 = 0$ and c, d are the roots of $x^2 + qx + 1 = 0$. Then find the value of $(a - c)(b - c)(a + d)(b + d)/(q^2 - p^2)$.

47. α, β are roots of the equation $\lambda(x^2 - x) + x + 5 = 0$. If λ_1 and λ_2 are the two values of λ for which the roots α, β are connected by the relation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = 4$, then the value of $\left(\frac{\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}}{14}\right)$ is

48. Let α, β be the roots of the equation $x^2 + ax + b = 0$ and γ, δ be the roots of $x^2 - ax + b - 2 = 0$. If $\alpha\beta\gamma\delta = 24$ and $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{5}{6}$, then find the value of a.

49. The least value of expression $x^2 + 2xy + 2y^2 + 4y + 7$ is:

50. If $a > b > 0$ and $a^3 + b^3 + 27ab = 729$ then the quadratic equation $ax^2 + bx - 9 = 0$ has roots α, β $(\alpha < \beta)$. Find the value of $4\beta - a\alpha$.

51. Let α and β be roots of $x^2 - 6(t^2 - 2t + 2)x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then find the minimum value of $\frac{a_{100}-2a_{98}}{a_{99}}$ (where $t \in R$)

PYQ's (Past Year Questions)

NATURE OF ROOTS, RELATION BETWEEN ROOTS AND COEFFICIENTS

1. For $0 < c < b < a$, let $(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ and $\alpha \neq 1$ be one of its root. Then, among the two statements **[31 Jan, 2024 (Shift-I)]**

   (I) If $\alpha \in (-1, 0)$, then b cannot be the geometric mean of a and c

   (II) If $\alpha \in (0, 1)$, then b may be the geometric mean of a and c

   (*a*) Both (I) and (II) are true
   (*b*) Neither (I) nor (II) is true
   (*c*) Only (II) is true
   (*d*) Only (I) is true

2. If α, β are the roots of the equation, $x^2 - x - 1 = 0$ and $S_n = 2023\alpha^n + 2024\beta^n$, then **[27 Jan, 2024 (Shift-II)]**

   (*a*) $2S_{12} = S_{11} + S_{10}$
   (*b*) $S_{12} = S_{11} + S_{10}$
   (*c*) $2S_{11} = S_{12} + S_{10}$
   (*d*) $S_{11} = S_{10} + S_{12}$

3. Let $x = \frac{m}{n}$ (m, n are co-prime natural numbers) be a solution of the equation $\cos(2\sin^{-1}x) = \frac{1}{9}$ and let $\alpha, \beta(\alpha > \beta)$ be the roots of the equation $mx^2 - nx - m + n = 0$. Then the point (α, β) lies on the line **[29 Jan, 2024 (Shift-II)]**

   (*a*) $3x + 2y = 2$
   (*b*) $5x - 8y = -9$
   (*c*) $3x - 2y = -2$
   (*d*) $5x + 8y = 9$

4. Let α, β be the roots of the equation $x^2 - \sqrt{6}x + 3 = 0$ such that $Im(\alpha) > Im(\beta)$. Let a, b be integers not divisible by 3 and n be a natural number such that $\frac{\alpha^{99}}{\beta}+\alpha^{98} = 3^n(a+ib), i = \sqrt{-1}$. Then $n + a + b$ is equal to ______.

   [29 Jan, 2024 (Shift-II)]

5. Let a, b, c be the length of three sides of a triangle satisfying the condition $(a^2 + b^2)x^2 - 2b(a + c) . x + (b^2 + c^2) = 0$. If the set of all possible values of x is the interval (α, β), then $12(\alpha^2 + \beta^2)$ is equal to ______.

   [31 Jan, 2024 (Shift-II)]

6. Let α, β be the roots of the equation $x^2 - x + 2 = 0$ with $Im(\alpha) > Im(\beta)$. Then $\alpha^6 + \alpha^4 + \beta^4 - 5\alpha^2$ is equal to ______.

   [29 Jan, 2024 (Shift-I)]

7. Let $\alpha, \beta \in N$ be roots of equation $x^2 - 70x + \lambda = 0$, where $\frac{\lambda}{2}, \frac{\lambda}{3} \in N$. If λ assumes the minimum possible value, then $\frac{\left(\sqrt{\alpha-1}+\sqrt{\beta-1}\right)(\lambda+35)}{|\alpha-\beta|}$ is equal to ______.

   [30 Jan, 2024 (Shift-I)]

8. Let α and β be the roots of the equation $px^2 + qx - r = 0$, where $p \neq 0$. If p, q and r be the consecutive terms of a non-constant G.P and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{3}{4}$, then the value of $(\alpha - \beta)^2$ is:

[01 Feb, 2024 (Shift-II)]

(a) $\frac{80}{9}$ (b) 9 (c) $\frac{20}{3}$ (d) 8

9. If 2 and 6 are the roots of the equation $ax^2 + bx + 1 = 0$, then the quadratic equation, whose roots are $\frac{1}{2a+b}$ and $\frac{1}{6a+b}$, is:

[04 April, 2024 (Shift-I)]

(a) $2x^2 + 11x + 12 = 0$ (b) $4x^2 + 14x + 12 = 0$

(c) $x^2 + 10x + 16 = 0$ (d) $x^2 + 8x + 12 = 0$

10. The number of distinct real roots of the equation $|x||x + 2| - 5|x + 1| - 1 = 0$ is ______.

[05 April, 2024 (Shift-I)]

11. Let, α, β be the distinct roots of the equation $x^2 - (t^2 - 5t + 6)x + 1 = 0$, $t \in R$ and $a_n = \alpha^n + \beta^n$. Then the minimum value of $\frac{a_{2023} + a_{2025}}{a_{2024}}$ is [06 April, 2024 (Shift-I)]

(a) $\frac{1}{4}$ (b) $-\frac{1}{2}$

(c) $-\frac{1}{4}$ (d) $\frac{1}{2}$

. Let α, β be roots of $x^2 + \sqrt{2}x - 8 = 0$. If $U_n = \alpha^n + \beta^n$, then $\frac{U_{10} + \sqrt{2}U_9}{2U_8}$ is equal to ______. [06 April, 2024 (Shift-II)]

13. Let α, β be the roots of the equation $x^2 + 2\sqrt{2}x - 1 = 0$. The quadratic equation, whose roots are $\alpha^4 + \beta^4$ and $\frac{1}{10}(\alpha^6 + \beta^6)$, is:

[09 April, 2024 (Shift-I)]

(a) $x^2 - 190x + 9466 = 0$ (b) $x^2 - 195x + 9466 = 0$

(c) $x^2 - 195x + 9506 = 0$ (d) $x^2 - 180x + 9506 = 0$

14. Let $\alpha, \beta; \alpha > \beta$, be the roots of the equation $x^2 - \sqrt{2}x - \sqrt{3} = 0$. Let $P_n = \alpha^n - \beta^n$, $n \in N$. Then $(11\sqrt{3} - 10\sqrt{2})P_{10} + (11\sqrt{2} + 10)P_{11} - 11P_{12}$ is equal to:

[09 April, 2024 (Shift-II)]

(a) $10\sqrt{2}P_9$ (b) $10\sqrt{3}P_9$

(c) $11\sqrt{2}P_9$ (d) $11\sqrt{3}P_9$

15. Let α, β be the roots of the quadratic equation $x^2 + \sqrt{6}x + 3 = 0$. Then $\frac{\alpha^{23} + \beta^{23} + \alpha^{14} + \beta^{14}}{\alpha^{15} + \beta^{15} + \alpha^{10} + \beta^{10}}$ is equal to

[12 April, 2023 (Shift-I)]

(a) 729 (b) 72

(c) 81 (d) 9

16. Let $\lambda \neq 0$ be a real number. Let α, β be the roots of the equation $14x^2 - 31x + 3\lambda = 0$ and α, γ be the roots of the equation $35x^2 - 53x + 4\lambda = 0$. Then $\frac{3\alpha}{\beta}$ and $\frac{4\alpha}{\gamma}$ are the roots of the equation: [29 Jan, 2023 (Shift-I)]

(a) $7x^2 + 245x - 250 = 0$

(b) $7x^2 - 245x + 250 = 0$

(c) $49x^2 - 245x + 250 = 0$

(d) $49x^2 + 245x + 250 = 0$

17. If a and b are the roots of equation $x^2 - 7x - 1 = 0$, then the value of $\frac{a^{21} + b^{21} + a^{17} + b^{17}}{a^{19} + b^{19}}$ is equal to

[11 April, 2023 (Shift-I)]

18. The minimum value of the sum of the squares of the roots of $x^2 + (3 - a)x + 1 = 2a$ is: [26 July, 2022 (Shift-II)]

(a) 4 (b) 5 (c) 6 (d) 8

19. Let p and q be two real numbers such that $p + q = 3$ and $p^4 + q^4 = 369$. Then $\left(\frac{1}{p} + \frac{1}{q}\right)^{-2}$ is equal to ______.

[26 June, 2022 (Shift-II)]

20. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{3a_9}$ is:

[25 Feb, 2021 (Shift-II)]

(a) 2 (b) 1 (c) 4 (d) 3

21. The number of real roots of the equation $e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^x + 1 = 0$ is:

[25 July, 2021 (Shift-I)]

(a) 1 (b) 4 (c) 2 (d) 6

22. The sum of all integral values of $k (k \neq 0)$ for which the equation $\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k}$ in x has no real roots, is

[26 Aug, 2021 (Shift-I)]

23. Let α and β be two real roots of the equation $(k + 1) \tan^2 x - \sqrt{2} \cdot \lambda \tan x = (1 - k)$, where $k (\neq -1)$ and λ are equal number. If $\tan^2(\alpha + \beta) = 50$, then a value of λ is

[07 Jan, 2020 (Shift-I)]

(a) $10\sqrt{2}$ (b) $5\sqrt{2}$ (c) 1 (d) 10

24. The least positive value of 'a' for which the equation, $2x^2 + (a - 10)x + \frac{33}{2} = 2a$ has real roots is ______.

[08 Jan, 2020 (Shift-I)]

Credit:--- Divansh.....

25. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of $ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$ is **[JEE Adv, 2020]**

(a) 0 (b) 8000

(c) 8080 (d) 16000

26. The number of integral values of m for which the equation $(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has no real root is: **[08 April, 2019 (Shift-II)]**

(a) infinitely many (b) 2

(c) 3 (d) 1

27. If one real root of the quadratic equation $81x^2 + kx + 256 = 0$ is cube of the other root, then a value of k is: **[11 Jan, 2019 (Shift-I)]**

(a) −81 (b) 100

(c) 144 (d) −300

28. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define $a_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, $n \geq 1$

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}, n \geq 2$. **[JEE Adv, 2019]**

Then which of the following options is/are correct?

(a) $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$ for all $n \geq 1$

(b) $\sum_{n=1}^{\infty} \dfrac{a_n}{10^n} = \dfrac{10}{89}$

(c) $\sum_{n=1}^{\infty} \dfrac{b_n}{10^n} = \dfrac{8}{89}$

(d) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

CONDITION FOR COMMON ROOTS

29. If the value of real number $\alpha > 0$ for which $x^2 - 5\alpha x + 1 = 0$ and $x^2 - \alpha x - 5 = 0$ have a common real roots is $\dfrac{3}{\sqrt{2\beta}}$ then β is equal to______. **[30 Jan, 2023 (Shift-II)]**

30. If for some $p, q, r \in R$, not all have same sign, one of the roots of the equation $(p^2 + q^2)x^2 - 2q(p + r)x + q^2 + r^2 = 0$ is also a root of the equation $x^2 + 2x - 8 = 0$, then $\dfrac{q^2 + r^2}{p^2}$ equal to ______ **[26 July, 2022 (Shift-I)]**

LOCATION OF ROOTS

31. The number of integral values of k, for which one root of the equation $2x^2 - 8x + k = 0$ lies in the interval $(1, 2)$ and its other root lies in the interval $(2, 3)$, is: **[01 Feb, 2023 (Shift-II)]**

(a) 2 (b) 0 (c) 1 (d) 3

32. Consider the two sets:

$A = \{m \in R$: both the roots of $x^2 - (m + 1)x + m + 4 = 0$ are real$\}$ and $B = [-3, 5)$. **[03 Sep, 2020 (Shift-I)]**

Which of the following is not true?

(a) $A \cap B = \{-3\}$ (b) $B - A = (-3, 5)$

(c) $A \cup B = R$ (d) $A - B = (-\infty, -3) \cup (5, \infty)$

33. The set of all real values of λ for which the quadratic equations, $(\lambda^2 + 1)x^2 - 4\lambda x + 2 = 0$ always have exactly one root in the interval $(0,1)$ is: **[03 Sep, 2020 (Shift-II)]**

(a) $(-3, -1)$ (b) $(2, 4]$

(c) $(0, 2)$ (d) $(1, 3]$

34. Consider the quadratic equation $(c - 5)x^2 - 2cx + (c - 4) = 0$, $c \neq 5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval $(0, 2)$ and its other root lies in the interval $(2, 3)$. Then the number of elements in S is: **[10 Jan, 2019 (Shift-I)]**

(a) 18 (b) 12

(c) 10 (d) 11

35. The number of integral values of m for which the quadratic expression, $(1 + 2m)x^2 - 2(1 + 3m)x + 4(1 + m)$, $x \in R$, is always positive, is: **[12 Jan, 2019 (Shift-II)]**

(a) 3 (b) 8

(c) 7 (d) 6

SPECIAL FORMS

36. The number of solutions, of the equation $e^{\sin x} - 2e^{-\sin x} = 2$ is **[31 Jan, 2024 (Shift-II)]**

(a) 2 (b) more than 2

(c) 1 (d) 0

37. The number of real solutions of the equation $x(x^2 + 3|x| + 5|x - 1| + 6|x - 2|) = 0$ is ______. **[30 Jan, 2024 (Shift-II)]**

38. Let the sum of the maximum and the minimum values of the function $f(x) = \dfrac{2x^2 - 3x + 8}{2x^2 + 3x + 8}$ be $\dfrac{m}{n}$, where $gcd(m, n) = 1$. Then $m + n$ is equal to: **[04 April, 2024 (Shift-I)]**

(a) 182 (b) 217 (c) 195 (d) 201

39. Let $S = \{\sin^2 2\theta : (\sin^4\theta + \cos^4\theta)x^2 + (\sin 2\theta)x + (\sin^6\theta + \cos^6\theta) = 0$ has real roots$\}$. If α and β be the smallest and largest elements of the set S, respectively, then $3((\alpha - 2)^2 + (\beta - 1)^2)$ equals____ **[04 April, 2024 (Shift-II)]**

40. The number of real solutions of the equation $x|x + 5| + 2|x + 7| - 2 = 0$ is ___________. **[05 April, 2024 (Shift-II)]**

41. Let x_1, x_2, x_3, x_4 be the solution of the equation

$4x^4 + 8x^3 - 17x^2 - 12x + 9 = 0$ and

$(4 + x_1^2)(4 + x_2^2)(4 + x_3^2)(4 + x_4^2) = \dfrac{125}{16} m$.

Then the value of m is _________. **[06 April, 2024 (Shift-I)]**

42. The sum of all the solutions of the equation $(8)^{2x} - 16 \cdot 8^x + 48 = 0$ is: [08 April, 2024 (Shift-I)]

(a) $1 + \log_6(8)]$

(b) $\log_8(6)$

(c) $1 + \log_8(6)]$

(d) $\log_8(4)$

43. The number of distinct real roots of the equation $|x + 1||x + 3| - 4|x + 2| + 5 = 0$, is ________.

[08 April, 2024 (Shift-II)]

44. The number of real solutions of the equation $3\left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) + 5 = 0$, is

[24 Jan, 2023 (Shift-II)]

(a) 4 (b) 0

(c) 3 (d) 2

45. Let $\lambda \in \mathbb{R}$ and let the equation E be $|x|^2 - 2|x| + |\lambda - 3| = 0$. Then the largest element in the set $S = (x + \lambda : x$ is an integer solution of $E)$ is .

[24 Jan, 2023 (Shift-I)]

46. Let $\alpha_1, \alpha_2, \ldots, \alpha_7$ be the roots of the equation $x^7 + 3x^5 - 13x^3 - 15x = 0$ and $|\alpha_1| \geq |\alpha_2| \geq \ldots \geq |\alpha_7|$.

Then $\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$ is equal to

[29 Jan, 2023 (Shift-II)]

47. $\frac{1}{(20-a)(40-a)} + \frac{1}{(40-a)(60-a)} + \ldots, + \frac{1}{(180-a)(200-a)} = \frac{1}{256}$ then the maximum value of a is: [29 July, 2022 (Shift-I)]

(a) 198 (b) 202 (c) 212 (d) 218

48. The sum of the cubes of all the roots of the equation $x^4 - 3x^3 - 2x^2 + 3x + 1 = 0$ is ... [26 June, 2022 (Shift-I)]

49. Let α, β $(\alpha > \beta)$ be the roots of the quadratic equation $x^2 - x - 4 = 0$. If $P_n = \alpha^n - \beta^n$, $n \in N$, then $\frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}P_{14}}$ is equal to.

[29 July, 2022 (Shift-II)]

50. The number of distinct real roots of the equation $x^5(x^3 - x^2 - x + 1) + x(3x^3 - 4x^2 - 2x + 4) - 1 = 0$ is________

[26 July, 2022 (Shift-I)]

51. Let a be an integer such that all the real roots of the polynomial $2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$ lie in the interval $(a, a + 1)$. Then $|a|$ is equal to_________.

[26 Feb, 2021 (Shift-II)]

52. Let $f(x)$ be a polynomial of degree 3 such that $f(k) = \frac{-2}{k}$ for $k = 2, 3, 4, 5$. Then the value of $52 - 10f(10)$ is equal to.

[01 Sep, 2021 (Shift-II)]

PW Challengers

NUMERICAL ANSWER TYPE QUESTIONS

1. Suppose a quadratic function $f(x) = ax^2 + bx + c$ $(a, b, c \in R$ and $a \neq 0)$ satisfies the following conditions:

(a) $f(x - 4) = f(2 - x)$ and $f(x) \geq x$.

(b) When $x \in (0, 2)$, $f(x) \leq \left(\frac{x+1}{2}\right)^2$.

(c) The minimum value of $f(x)$ on R is 0

Then the maximal $m(m > 1)$ such that there exists $t \in R$, $f(x + t) \leq x$ holds so long as $x \in [1, m]$, is

2. Let $A = \{x | x^2 - 4x + 3 < 0, x \in R\}$, $B = \{x | 2^{1-x} + a \leq 0, x^2 - 2(a + 7)x + 5 \leq 0, x \in R\}$. If $A \subseteq B$, then the absolute value of the sum of maximum and minimum value of real number a is_______.

3. If the expression $2x^2 + 3xy + y^2 + 2y + 3x + \lambda$ after factorizing into two linear factors is equivalent to $(\alpha x + \beta y + \gamma)(\phi x + \psi y + \xi)$, $\alpha > \phi$ then the absolute value of $(\alpha + \beta + \gamma) - \lambda(\phi + \psi + \xi) =$

SINGLE CORRECT TYPE QUESTIONS

4. Let p be an odd prime. Let k be a positive integer such that $\sqrt{k^2 - pk}$ is also a positive integer. Then $k =$ ________.

(a) $(p + 1)^4$ (b) $\frac{(p+1)^2}{4}$

(c) $4(p + 1)^4$ (d) $\frac{(p+1)}{4}$

5. There are real numbers a, b and c and a positive number λ such that $f(x) = x^3 + ax^2 + bx + c$, if $f(x) = 0$ has three real roots x_1, x_2 and x_3 satisfying

(1) $x_2 - x_1 = \lambda$ (2) $x_3 > \frac{1}{2}(x_1 + x_2)$.

Find the maximum value of $\frac{2a^3 + 27c - 9ab}{\lambda^3}$.

(a) $-\frac{2}{3}\sqrt{3}$ (b) $\frac{2}{3}\sqrt{3}$ (c) $\frac{3}{2}\sqrt{3}$ (d) $\frac{1}{3}\sqrt{2}$

6. Find all polynomials $P(x)$ with real coefficients, which satisfy the equation $P(a-b)+P(b-c)+P(c-a)=2P(a+b+c)$ for all real numbers a, b, c such that $ab+bc+ca=0$.

7. The number of ordered pairs (x, y) of positive integers satisfying the equation $x^2+y^2=13(x+y)$ is

(a) 6 (b) 5 (c) 4 (d) 3

8. If $a, b, c, d \in R$ satisfies $5(a^2+b^2+c^2+d^2-ab-bc-cd-d)+2=0$ then

	Column-I		Column-II
A.	$c=$	p.	$\frac{d}{2}$
B.	$b=$	q.	$\frac{c}{3}$
C.	$d=$	r.	$a+b$
D.	$a=$	s.	$a+c$

(a) A → r, B → p, C → s, D → q

(b) A → p, B → r, C → q, D → q

(c) A → p, B → r, C → q, D → s

(d) A → p, B → q, C → r, D → q

9. If the equation $x^4+x^3-2ax^2-ax+a^2=0$ has 4 real roots then the least possible value of 'a' is

(a) $\frac{2}{4}$ (b) $-\frac{2}{4}$

(c) 0 (d) $\frac{1}{4}$

10. Mathematician J.Sachin was working on variables $x, y, z \in R$ during his work he found an interesting equation $2x^2+y^2+2z^2-8x+2y-2xy+2xz-16z+35=0$. Then $x+y+z=$

(a) 0 (b) 1

(c) 4 (d) N.O.T

11. S. Pankaj solved the equation to evaluate pair of solution of

$$5x\left(1+\frac{1}{x^2+y^2}\right)=12$$

$$5y\left(1-\frac{1}{x^2+y^2}\right)=4$$

as (α, β) & (γ, δ), $\alpha<\gamma$ then find $5(\alpha+\beta)+\gamma+\delta$

(a) −2 (b) 4

(c) 1 (d) −1

ANSWER KEY

CONCEPT APPLICATION

1. (a) 2. (a) 3. (c) 4. (b) 5. (b) 6. (b) 7. (d) 8. (c) 9. (d) 10. (d)
11. (b) 12. (b) 13. (c) 14. (d) 15. (a) 16. (d) 17. (a) 18. (b) 19. (d) 20. (a)
21. (a) 22. (b)
23. (i) $\left[\frac{3}{4},\infty\right)$ (ii) $\left[\frac{31}{8},\infty\right)$ (iii) $[-\infty,8)$ (iv) $\left[-\frac{1}{4},8\right)$
(v) $\left[-1-\frac{2}{\sqrt{3}},-1+\frac{2}{\sqrt{3}}\right]$ (vi) $\left[1-\frac{2}{\sqrt{3}},1+\frac{2}{\sqrt{3}}\right]$ (vii) $\left(-\infty,3-\sqrt{8}\right]\cup\left[3+\sqrt{8},\infty\right)$
(viii) $\left(-\infty,-7-\sqrt{48}\right]\cup\left[-7+\sqrt{48},\infty\right)$ (ix) $(-\infty,\infty)$ (x) $\left[-\frac{5}{3},\frac{1}{2}\right)$ (xi) $[1,\infty)$ (xii) $[4,56]$
24. (b) 25. (c) 26. (a) $[9, 10)$ (b) $[9,\infty)$ (c) $m<0$ (d) ϕ (e) ϕ (f) ϕ
27. (a) 28. (c) 29. (a) 30. (d) 31. (a) 32. (a)

SCHOOL LEVEL PROBLEMS

1. (c) 2. (b) 3. (c) 4. (a) 5. (d) 6. (b) 7. (b) 8. (b) 9. (a) 22. (a)
23. (b) 24. (b) 25. (d)

PRARAMBH (TOPICWISE)

1. (c) 2. (d) 3. (a) 4. (c) 5. (d) 6. (b) 7. (a) 8. (c) 9. (a) 10. (c)
11. (b) 12. (c) 13. (c) 14. (a) 15. (c) 16. (b) 17. (a) 18. (b) 19. (c) 20. (d)
21. (c) 22. (c) 23. (d) 24. (b) 25. (c) 26. (c) 27. (d) 28. (b) 29. (b) 30. (b)

Credit:--- Divansh.....

31. (*c*) 32. (*a*) 33. (*a*) 34. (*c*) 35. (*b*) 36. (*b*) 37. (*b*) 38. (*b*) 39. (*d*) 40. (*c*)
41. (*c*) 42. (*b*) 43. (*b*) 44. (*d*) 45. (*b*) 46. (*a*) 47. (*d*) 48. (*a*) 49. (*d*) 50. (*b*)

PRABAL (JEE MAIN LEVEL)

1. (*c*) 2. (*a*) 3. (*a*) 4. (*b*) 5. (*d*) 6. (*c*) 7. (*a*) 8. (*a*) 9. (*a*) 10. (*c*)
11. (*a*) 12. (*a*) 13. (*d*) 14. (*c*) 15. (*c*) 16. (*b*) 17. (*c*) 18. (*a*) 19. (*c*) 20. (*a*)
21. (*c*) 22. (*b*) 23. (*a*) 24. (*c*) 25. (*a*) 26. (*a*) 27. (*a*) 28. (*b*) 29. (*c*) 30. (*a*)
31. (*b*) 32. (*c*) 33. (*d*) 34. (*a*) 35. (*d*) 36. (*b*) 37. (*d*) 38. (*d*) 39. (*c*) 40. [167]
41. [99] 42. [2] 43. [8] 44. [9] 45. [23] 46. [324] 47. [191] 48. [0] 49. [2] 50. [3]

PARIKSHIT (JEE ADVANCED LEVEL)

1. (*a*) 2. (*d*) 3. (*c*) 4. (*c*) 5. (*a*) 6. (*a*) 7. (*c*) 8. (*a*) 9. (*a*) 10. (*b*)
11. (*c*) 12. (*d*) 13. (*a*) 14. (*a,b,c*) 15. (*a,d*) 16. (*a,d*) 17. (*a,b,c,d*) 18. (*b,c,d*) 19. (*b,c,d*) 20. (*a,b*)
21. (*d*) 22. (*c*) 23. (*c*) 24. (*c*) 25. (*a*) 26. (*c*) 27. (*c*) 28. (*b*) 29. (*d*) 30. (*d*)
31. (*c*) 32. (*b*) 33. (*d*) 34. (*c*) 35. (*b*) 36. (*a*) 37. (*c*) 38. (*b*) 39. (*a*) 40. (*b*)
41. (*c*) 42. (*b*) 43. [2] 44. [8] 45. [11] 46. [1] 47. [73] 48. [10] 49. [3] 50. [13]
51. [6]

PYQ's (PAST YEAR QUESTIONS)

1. (*a*) 2. (*b*) 3. (*d*) 4. [49] 5. [36] 6. [13] 7. [60] 8. (*a*) 9. (*d*) 10. [3]
11. (*c*) 12. [4] 13. (*c*) 14. (*b*) 15. (*c*) 16. (*c*) 17. [51] 18. (*c*) 19. [4] 20. (*a*)
21. (*c*) 22. [66] 23. (*d*) 24. [8] 25. (*d*) 26. (*a*) 27. (*d*) 28. (*a,b,c*) 29. [13] 30. [272]
31. (*c*) 32. (*d*) 33. (*d*) 34. (*d*) 35. (*c*) 36. (*d*) 37. [1] 38. (*d*) 39. [4] 40. [3]
41. [221] 42. (*c*) 43. [2] 44. (*b*) 45. [5] 46. [9] 47. (*c*) 48. [36] 49. [16] 50. [3]
51. [2] 52. [26]

PW CHALLENGERS

1. [9] 2. [5] 3. [1] 4. (*b*) 5. (*c*) 6. $[\alpha x^4+\beta x^2]$ 7. (*b*) 8. (*a*) 9. (*c*) 10. (*a*)
11. (*b*)

VISUALIZATION EXERCISE

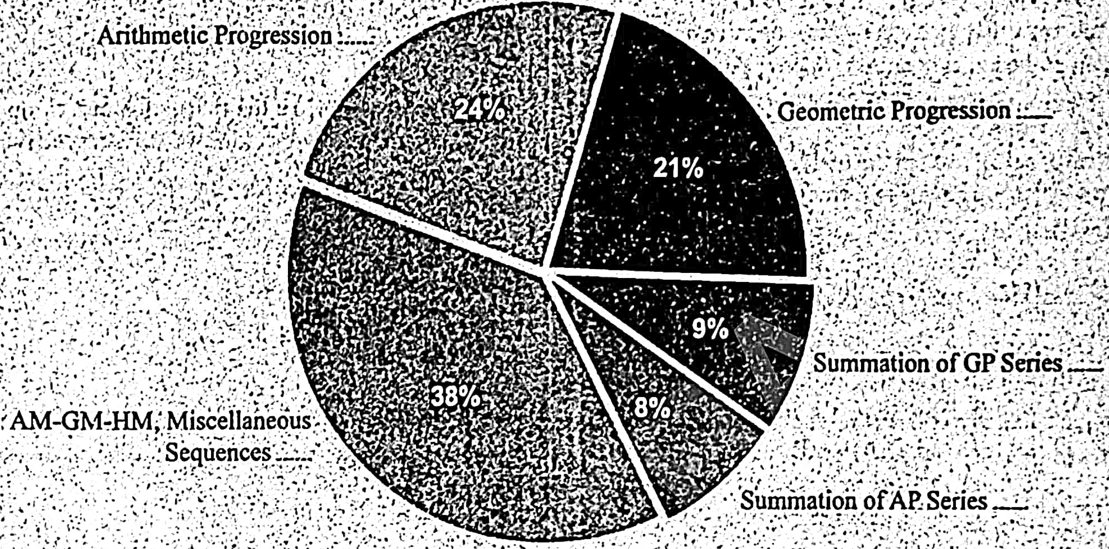
- Spend 10 minutes daily visualizing your success in the exam. Picture yourself walking into the exam hall with confidence, solving questions with ease.
- Focus on the emotions of pride and accomplishment to foster a positive outlook and build confidence in achieving your goals.

Credit:--- Divansh.....

CHAPTER 6

Sequences and Series

Topicwise Weightage of JEE Main 6 Years Paper (124 Sets)



"How's the Josh?" for these Topics: *Mark your confidence level in the blank space around the topic (Low-L, Medium-M, High-H)*

INTRODUCTION

A **sequence** is a set of terms arranged in a definite order with a rule for obtaining the terms.

e.g. 1, 1/2, 1/3, ..., $1/n$, ... is a sequence.

A sequence is a function whose domain is the set N of natural numbers. Since the domain for every sequence is the set of N natural numbers, therefore a sequence is represented by its range. $f: N \to R$, then $f(n) = t_n$, $n \in N$ is called a sequence and is denoted by

$$\{f(1), f(2), f(3), ...\}$$
$$= \{t_1, t_2, t_3, ...\} = \{t_n\}$$

Real Sequence: A sequence whose all terms are real is called a real sequence.

Examples:

(*i*) 2, 5, 8, 11, ...

(*ii*) 4, 1, –2, –5, ...

(*iii*) 3, –9, 27, –81, ...

Types of Sequence: On the basis of the number of terms there are two types of sequence.

(*i*) **Finite sequence:** A sequence is said to be finite if it has finite number of terms.

(*ii*) **Infinite sequence:** A sequence is said to be infinite if it has infinite number of terms.

Series: By adding or substracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, ..., a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + ... + a_n$ is a series.

Example:

(*i*) $1+2+3+4+ .... + n$

(*ii*) $2+4+8+16+ ....$

Progression: It is not necessary that the terms of a sequence always described by some explicit formula of the n^{th} term. Those sequences whose terms follow certain patterns are called progressions.

ARITHMETIC PROGRESSION (AP)

AP is a sequence whose consecutive terms increase or decrease by a fixed number. This fixed number is called the **common difference.** If a is the first term & d the common difference, then AP can be written as $a, a+d, a+2d, ... a+(n-1)d, ...$ **general term (n^{th} term)** of this AP, $t_n = a + (n-1)d$, where $d = a_n - a_{n-1}$.

The **sum of the first n terms** of the AP is given by;

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l].$$

where l is the last term.

Arithmetic Mean: If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c.

Credit:--- Divansh.....

So, A.M. of two numbers a and b is $\frac{a+b}{2}$.

AM for any n positive number $a_1, a_2, ..., a_n$ is;

$$A = \frac{a_1+a_2+a_3+.....+a_n}{n}.$$

General Properties

- (*i*) Three numbers in AP can be taken as $a-d, a, a+d$; four numbers in AP can be taken as $a-3d, a-d, a+d, a+3d$; five numbers in AP are $a-2d, a-d, a, a+d, a+2d$ & six terms in AP are $a-5d, a-3d, a-d, a+d, a+3d, a+5d$ etc.
- (*ii*) The common difference can be zero, positive or negative.
- (*iii*) The sum of the terms in an AP which are equidistant from the beginning & end is constant and equal to the sum of first & last terms.
- (*iv*) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it.
  $a_n = 1/2\,(a_{n-k} + a_{n+k}), k < n.$
  For $k = 1, a_n = (1/2)(a_{n-1} + a_{n+1})$;
  For $k = 2, a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on.
- (*v*) $t_r = S_r - S_{r-1}$
- (*vi*) If $a_1, a_2, a_3, ..., a_n$ are in A.P., then
  - (*a*) $a_1 + k, a_2 + k, a_3 + k, ... a_n + k$ are also in A.P.
  - (*b*) $ka_1, ka_2, ..., ka_n$ are also in A.P.
  - (*c*) $\frac{a_1}{k}, \frac{a_2}{k}, ..., \frac{a_n}{k}, k \neq 0$ are also in A.P.
- (*vii*) If $a_1, a_2, a_3, ...$, and $b_1, b_2, b_3, ...$, are two A.Ps., then
  - (*a*) $a_1 + b_1, a_2 + b_2, a_3 + b_3, ...$, are also in A.P.
  - (*b*) $a_1 - b_1, a_2 - b_2, a_3 - b_3, ...$, are also in A.P.
- (*viii*) If a_1, a_2, a_3, a_n, are in A.P., then
  - (*a*) $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ... = 2a_1 + (n-1)d$
  - (*b*) $a_r = \frac{a_{r-k} + a_{r+k}}{2}, 0 \leq k \leq n-r$
- (*ix*) If n^{th} term of a sequence is a linear expression in n then the sequence is an A.P.
- (*x*) If the sum of first n terms of a sequence is a quadratic expression in n, then the sequence is an A.P.
- (*xi*) Three numbers a, b, c are in A.P. if and only if $b - a = c - b$, i.e., if and only if $a + c = 2b$.

Train Your Brain

Example 1: If t_{54} of an A.P. is -61 and $t_4 = 64$, find t_{10}.

Sol. Let a be the first term and d be the common difference

so $t_{54} = a + 53d = -61$ ... (*i*)

and $t_4 = a + 3d = 64$ ... (*ii*)

equation (*i*) – (*ii*) we get

$\Rightarrow 50d = -125$

$$d = -\frac{5}{2} \Rightarrow a = \frac{143}{2} \text{ so } t_{10} = \frac{143}{2} + 9\left(-\frac{5}{2}\right) = 49$$

Example 2: Find the number of terms in the sequence 4, 12, 20, ..., 108.

Sol. $a = 4, d = 8$ so $108 = 4 + (n-1)8 \Rightarrow n = 14$

Example 3: Find the sum of all the three digit natural numbers which on division by 7 leaves remainder 3.

Sol. All these numbers are 101, 108, 115,... 997, to find n.

$997 = 101 + (n-1)\,7$

$$\Rightarrow n = 129 \text{ so } S = \frac{129}{2}[101 + 997] = 70821.$$

Example 4: The sum of n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11^{th} terms.

Sol. Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively, then

$$\frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11^{th} terms

Put $\frac{n-1}{2} = 10 \Rightarrow n = 21$ on both sides

$$\text{so, } \frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$$

Example 5: The sum of first three terms of an A.P. is 27 and the sum of their squares is 293. Find the sum to 'n' terms of the A.P.

Sol. Let $a - d, a, a + d$ be the numbers $\Rightarrow a = 9$

Also $(a-d)^2 + a^2 + (a+d)^2 = 293$.

$\Rightarrow 3a^2 + 2d^2 = 293 \Rightarrow d^2 = 25 \Rightarrow d = \pm 5$

therefore numbers are 14, 9, 4.

Hence the A.P. is 4, 9, 14, ... or 14, 9, 4, ...

$$\Rightarrow s_n = \frac{n}{2}[5n + 3] \text{ or } s_n = \frac{n}{2}[33 - 5n]$$

Example 6: If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P., prove that a^2, b^2, c^2 are also in A.P.

Sol. $\because \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$$\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\Rightarrow \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\Rightarrow \frac{b-a}{b+c} = \frac{c-b}{a+b} \Rightarrow b^2 - a^2 = c^2 - b^2$$

$\Rightarrow a^2, b^2, c^2$ are in A.P.

Concept Application

1. Find the sum of all odd numbers between 1 and 1000 which are divisible by 3
2. The ratio between the sum of n term of two A.P.'s is $3n + 8 : 7n + 15$. Then find the ratio between their 12th term
3. If sum of n terms of a sequence is given by $S_n = 3n^2 - 4n$, find its 50th term.
4. If $a(b + c)$, $b(c + a)$, $c(a + b)$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.
5. If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference $\neq 0$. then find the value of $\sum_{i=1}^{5} a_i$, when $a_3 = 2$.
6. Write down the sequence whose n^{th} term is
   (i) $\frac{2^n}{n}$
   (ii) $\frac{3+(-1)^n}{3^n}$

GEOMETRIC PROGRESSION (GP)

GP is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the proceeding terms multiplied by same constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the G.P. Therefore $a, ar, ar^2, ar^3, ar^4, \ldots$ is a GP with a as the first term & r as common ratio.

(*i*) **General term** of a G.P. $= ar^{n-1}$

(*ii*) **Sum to n terms**

Let a is first term and r is common ratio of G.P. then sum is given by

$$S_n = \frac{a(1-r^n)}{1-r} \text{ provided } r \neq 1$$

(*iii*) **Sum of an infinite GP** when $|r| < 1$ when $n \to \infty$ $r^n \to 0$

if $|r| < 1$ therefore, $S_\infty = \frac{a}{1-r}, (|r|<1)$.

Geometric Means: If a, b, c are in GP, b is the GM between a & c then $b^2 = ac$,

therefore $b = \sqrt{ac}$; $a > 0, c > 0$.

Geometric means of n positive numbers $a_1, a_2, \ldots a_n$ is $(a_1.a_2 \ldots a_n)^{1/n}$

General Properties of Geometrical Progression

(*i*) $a_1, a_2, a_3, \ldots$ are in G.P. then
$a_1k, a_2k, a_3k, \ldots$ and $a_1/k, a_2/k, a_3/k \ldots$ are also in G.P. $(k \neq 0)$

(*ii*) If $a_1, a_2, a_3, \ldots$ are in G.P. Then $1/a_1, 1/a_2, 1/a_3, \ldots$ are also in G.P.

(*iii*) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two G.P.s, then $a_1b_1, a_2b_2, a_2b_2, a_3b_3, \ldots$ and $a_1/b_1, a_2/b_2, a_3/b_3, \ldots$ are also in G.P.

(*iv*) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are tow G.P.s, then $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$ are not in G.P.

(*v*) If $a_1, a_2, a_3, \ldots$ are in G.P. $(a_i > 0, \forall i)$, then $\log a_1, \log a_2, \log a_3, \ldots$ are in A.P. In this case the converse also holds good.

(*vi*) If $a_1, a_2, a_3, \ldots a_n$ are in G.P., then

(*a*) $a_1a_n = a_2a_{n-1} = a_3a_{n-2} = \ldots = a^2r^{n-1}$

(*b*) $a_r = \sqrt{a_{r-k}a_{r+k}}, \quad 0 \le k \le n-r$

(*vii*) If $a_1, a_2, a_3, a_4 \ldots a_{n-1}\ a_n$ are in G.P., then

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = \frac{a_n}{a_{n-1}}$$

$\Rightarrow a_2^2 = a_3a_1, a_3^2 = a_2a_4, \ldots$

also $a_2 = a_1r, a_3 = a_1r^2, a_4 = a_1r^3, \ldots, a_n = a_1r^{n-1}$

where r is the common ratio.

(*viii*) Three numbers in G.P. can be taken as $\frac{a}{r}, a, ar$; Five numbers in G.P. can be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$ etc.

In general: $(2m + 1)$ numbers in G.P. can be written as $(m \in N)$

$$\frac{a}{r^m}, \frac{a}{r^{m-1}}, \ldots, \frac{a}{r}, a, ar, \ldots, ar^{m-1}, ar^m$$

(*ix*) Four numbers in G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$;

Six numbers in G.P. can be taken as $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$; etc.

In **general:** $(2m)$ numbers in G.P. can be written as $(m \in N)$

$$\frac{a}{r^{2m-1}}, \frac{a}{r^{2m-3}}, \ldots, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, \ldots, ar^{2m-3}, ar^{2m-1}$$

Train Your Brain

Example 7: If the first term of G.P. is 7, its n^{th} term is 448 and sum of first n terms is 889, then find the fifth term of G.P.

Sol. Given a = 7 the first term, $t_n = ar^{n-1} = 7(r)^{n-1} = 448$

$\Rightarrow 7r^n = 448\,r$

Also $S_n = \frac{a(r^n-1)}{r-1} = \frac{7(r^n-1)}{r-1}$

$\Rightarrow 889 = \frac{448r-7}{r-1} \Rightarrow r = 2.$

Hence $T_5 = ar^4 = 7(2)^4 = 112$

Example 8: Find three numbers in G.P. having sum 19 and product 216.

Sol. Let the three numbers be $\frac{a}{r}, a, ar$

so $a\left[\frac{1}{r}+1+r\right] = 19$ ... (*i*)

and $a^3 = 216 \Rightarrow a = 6$

so from (*i*) $6r^2 - 13r + 6 = 0. \Rightarrow r = \frac{3}{2}, \frac{2}{3}$

Hence the three numbers are 4, 6, 9.

Example 9: Evaluate 7 + 77 + 777 + ... upto n terms.

Sol. Let $S = 7 + 77 + 777 + ...$upto n terms.

$$= \frac{7}{9}[9 + 99 + 999 + ...]$$

$$= \frac{7}{9}[(10-1) + (10^2-1) + (10^3-1) + ... + \text{ upto } n \text{ terms}]$$

$$= \frac{7}{9}[10 + 10^2 + 10^3 + ... + 10^n - n]$$

$$= \frac{7}{9}\left(\frac{10(10^n-1)}{9} - n\right) = \frac{7}{81}[10^{n+1} - 9n - 10]$$

Example 10: If the third and fourth terms of an arithmetic sequence are increased by 3 and 8 respectively, then the first four terms form a geometric sequence. Find

(*i*) the sum of the first four terms of A.P.

(*ii*) second term of the G.P.

Sol. $a, (a+d), (a+2d), (a+3d)$ in A.P.

$a, a+d, (a+2d+3), (a+3d+8)$ are in G.P.

hence $a + d = ar$

also $r = \frac{a+d}{a} = \frac{a+2d+3}{a+d} = \frac{a+3d+8}{a+2d+3}$ $\therefore$ $\frac{d+3}{d} = \frac{d+5}{d+3}$

$\Rightarrow d^2 + 6d + 9 = d^2 + 5d$

$\Rightarrow d = -9$

$\therefore \frac{a-9}{a} = \frac{a-15}{a-9}$

$\Rightarrow a^2 - 18a + 81 = a^2 - 15a$

$\Rightarrow 3a = 81 \Rightarrow a = 27$

hence A.P. is 27, 18, 9, 0, G.P. is 27, 18, 12, 8

(*i*) sum of the first four terms of A.P. = 54

(*ii*) 2nd term of G.P. = 18

Concept Application

7. Find the G.P. if the common ratio of G.P. is 3, n^{th} term is 486 and sum of first n terms is 728.
8. If $x, 2y, 3z$ are in A.P. where the distinct numbers x, y, z are in G.P. Then find the common ratio of G.P
9. Three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. The common ratio of G.P. is:
10. Evaluate 9 + 99 + 999 + ...upto n terms.

HARMONIC PROGRESSION (HP)

A sequence is said to be HP if the reciprocals of its terms are in AP.

If the sequence $a_1, a_2, a_3, ..., a_n$ is an HP then $1/a_1, 1/a_2, ..., 1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. For HP whose first term is a & second term is b, the n^{th} term is $t_n = \frac{ab}{b+(n-1)(a-b)}$. If a, b, c are in HP

$$\Rightarrow b = \frac{2ac}{a+c} \text{ or } \frac{a}{c} = \frac{a-b}{b-c}.$$

Harmonic Mean: If a, b, c are in HP, b is the HM between a & c, then $b = 2ac/[a+c]$.

If a, b are two given number such that

$a, H_1, H_2, H_3 ... H_n, b$ are in H.P.

then $H_1, H_2, H_3 ... H_n$, are n harmonic means between a and b.

$\because a, H_1, H_2, H_3 ... H_n, b \rightarrow$ H.P.

$\therefore \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3} ... \frac{1}{H_n}, \frac{1}{b} \rightarrow$ A.P.

Common difference $(d) = \frac{\frac{1}{b} - \frac{1}{a}}{n+1}$

So, $\frac{1}{H_1} = \frac{1}{a} + d, \frac{1}{H_2} = \frac{1}{a} + 2d, ... \frac{1}{H_n} = \frac{1}{a} + nd$

Train Your Brain

Example 11: If $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$ prove that a, b, c are in HP unless $b = a + c$.

Sol. $\Rightarrow \frac{a+c}{ac} + \frac{c-b+a-b}{(a-b)(c-b)} = 0$

$\Rightarrow \frac{a+c}{ac} + \frac{(a+c)-2b}{ac-b(a+c)+b^2} = 0$

Let $a + c = \lambda$ $\therefore \frac{\lambda}{ac} + \frac{\lambda - 2b}{ac - b\lambda + b^2} = 0$

$\Rightarrow \frac{ac\lambda - b\lambda^2 + b^2\lambda + ac\lambda - 2abc}{ac(ac - b\lambda + b^2)} = 0$

$\Rightarrow ac\lambda - b\lambda^2 + b^2\lambda + ac\lambda - 2abc = 0$

$\Rightarrow 2ac(\lambda - b) - b\lambda(\lambda - b) = 0$

$\Rightarrow (2ac - b\lambda)(\lambda - b) = 0$

$\Rightarrow \lambda = b$ or $\lambda = \frac{2ac}{b}$

$\Rightarrow a + c = b$ or $a + c = \frac{2ac}{b}$ $(\because a + c = \lambda)$

$\Rightarrow a + c = b$ or $b = \frac{2ac}{a+c}$

$\Rightarrow a, b, c$ are in H.P. or $a + c = b$.

Example 12: If p^{th}, q^{th}, r^{th} terms of an H.P. be a, b, c respectively, prove that $(q-r)bc+(r-p)ac+(p-q)ab=0$

Sol. Let 'x' be the first term and 'd' be the common difference of the corresponding A.P..

so $\frac{1}{a}=x+(p-1)d$ ... (*i*)

$\frac{1}{b}=x+(q-1)d$ ... (*ii*)

$\frac{1}{c}=x+(r-1)d$ ... (*iii*)

$(i)-(ii) \Rightarrow ab(p-q)d=b-a$ ... (*iv*)

$(ii)-(iii) \Rightarrow bc(q-r)d=c-b$ ... (*v*)

$(iii)-(i) \Rightarrow ac(r-p)d=a-c$ ... (*vi*)

$(iv)+(v)+(vi)$ gives

$bc(q-r)+ac(r-p)+ab(p-q)=0.$

Concept Application

11. a, b, c are in H.P. then prove that $\frac{b+a}{b-a}+\frac{b+c}{b-c}=2$
12. The 7th term of a H.P. is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$, find the 20th term of H.P.
13. If a, b, c, d are in H.P., then find the value of $ab+bc+cd$.

INSERTION OF MEANS BETWEEN TWO NUMBER

n - Arithmetic Means Between Two Numbers:

If a, b are any two given numbers & $a, A_1, A_2, \ldots, A_n, b$ are in AP then $A_1, A_2, \ldots A_n$ are the n AM's between a & b.

$$A_1=a+d=a+\frac{b-a}{n+1},$$

$$A_2=a+2d=a+\frac{2(b-a)}{n+1}$$

$, \ldots\ldots,$

$$A_n=a+nd=a+\frac{n(b-a)}{n+1}$$

Note: Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r=nA$ where A is the single AM between a & b.

n-Geometric Means Between a, b: If a, b are two given numbers & $a, G_1, G_2, \ldots, G_n, b$ are in GP. Then $G_1, G_2, G_3, \ldots, G_n$ are n GM's between a & b.

$G_1=ar=a(b/a)^{1/(n+1)}$, $G_2=ar^2=a(b/a)^{2/(n+1)}$, ..., $G_n=ar^n=a(b/a)^{n/(n+1)}$

Note: The product of n GM's between a & b is equal to the n^{th} power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r=(G)^n$ where G is the single GM between a & b.

for any two positive numbers a, b

arithmetic mean = $AM \Rightarrow A=\frac{a+b}{2}$

Geometric mean = $GM \Rightarrow G=\sqrt{ab}$

Harmonic mean = $HM \Rightarrow H=\frac{2ab}{a+b}$

$$A \times H=\left(\frac{a+b}{2}\right)\frac{2ab}{a+b}=ab=\left(\sqrt{ab}\right)^2=G^2$$

INEQUALITIES

(*i*) **A.M. ≥ G.M. ≥ H.M.**

Let $a_1, a_2, \ldots\ldots\ldots, a_n$ be n positive real numbers, then we define their arithmetic mean

(A), geometric mean (G) and harmonic mean (H) as

$$A=\frac{a_1+a_2+\ldots\ldots+a_n}{n},$$

$$G=(a_1a_2\ldots\ldots a_n)^{1/n} \text{ and } H=\frac{n}{\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots\ldots\frac{1}{a_n}\right)}$$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if

$a_1=a_2=\ldots\ldots=a_n$

Note: $x+\frac{1}{x} \geq 2$, if $x>0$ and $x+\frac{1}{x} \leq -2$, if $x<0$.

(*ii*) **Weighted Means**

Let $a_1, a_2, \ldots\ldots, a_n$ be n positive real numbers and $w_1, w_2, \ldots\ldots, w_n$ be n positive rational numbers. Then we define weighted Arithmetic mean (A^*), weighted Geometric mean (G^*) and weighted harmonic mean (H^*) as

$$A^*=\frac{a_1w_1+a_2w_2+\ldots+a_nw_n}{w_1+w_2+\ldots+w_n},$$

$$G^*=(a_1^{w_1}a_2^{w_2}\ldots a_n^{w_n})^{\frac{1}{w_1+w_2+\ldots+w_n}}$$

$$\text{and } H^*=\frac{w_1+w_2+\ldots+w_n}{\frac{w_1}{a_1}+\frac{w_2}{a_2}+\ldots+\frac{w_n}{a_n}}.$$

$A^* \geq G^* \geq H^*$ More over equality holds at either place if & only if $a_1=a_2\ldots=a_n$

(*iii*) **Cauchy's Schwartz Inequality:**

If $a_1, a_2 \ldots a_n$ and $b_1, b_2 \ldots b_n$ are $2n$ real numbers, then $(a_1b_1+a_2b_2+\ldots+a_nb_n)^2 \leq (a_1^2+a_2^2+\ldots+a_n^2)(b_1^2+b_2^2+\ldots+b_n^2)$ with the equality holding if and only if $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots\ldots=\frac{a_n}{b_n}$.

Credit:--- Divansh.....

(iv) Arithmetic Mean of M^th Power

Let $a_1, a_2, ..., a_n$ be n positive real numbers and let m be a real number, then

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} \geq \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m,$$

if $m \in R - [0, 1]$

However if $m \in (0, 1)$, then

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} \leq \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m$$

Obviously if $m \in \{0, 1\}$, then

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} = \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m$$

Train Your Brain

Example 13: Insert 4 H.M between $\frac{2}{3}$ and $\frac{2}{13}$.

Sol. Let 'd' be the common difference of corresponding A.P..

$$\text{So } d = \frac{\frac{13}{2} - \frac{3}{2}}{5} = 1. \therefore \frac{1}{H_1} = \frac{3}{2} + 1 = \frac{5}{2} \text{ or } H_1 = \frac{2}{5}$$

$$\frac{1}{H_2} = \frac{3}{2} + 2 = \frac{7}{2} \text{ or } H_1 = \frac{2}{7}$$

$$\frac{1}{H_3} = \frac{3}{2} + 3 = \frac{9}{2} \text{ or } H_3 = \frac{2}{9}$$

$$\frac{1}{H_4} = \frac{3}{2} + 4 = \frac{11}{2} \text{ or } H_4 = \frac{2}{11}$$

Example 14: If $a, b, c > 0$, prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$

Sol. Using the relation A.M. ≥ G.M. we have

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq \left(\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}\right)^{\frac{1}{3}} \Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

Example 15: If one AM 'a' and two GM's p and q be inserted between any two given numbers then show that $p^3 + q^3 = 2\,apq$.

Sol. Let two numbers are A and B.

If one AM 'a' is inserted between A & B then A, a, B .....are in A.P.

therefore $a = \frac{A+B}{2}$ or $\boxed{A + B = 2a}$ ... (*i*)

If two GM's p & q are inserted between A & B then A, p, q, B ..... are in G.P.

therefore $p^2 = Aq$ & $q^2 = Bp$

or $\boxed{A = \frac{p^2}{q}}$ ... (*ii*) or $\boxed{B = \frac{q^2}{p}}$ ... (*iii*)

Adding equation (ii) & (iii)

$A + B = \frac{p^2}{q} + \frac{q^2}{p}$ or $2a = \frac{p^3 + q^3}{pq}$ (using equation (*i*))

or $\boxed{p^3 + q^3 = 2apq}$

Example 16: Insert 10 A.M. between 3 and 80.

Sol. Here 3 is the first term and 80 is the 12th term of A.P. so

$80 = 3 + 11d$

$\Rightarrow d = 7$

So the series is 3,10,17,24,...,73. 80

$\therefore$ Required means are 10, 17, 24,...,73.

Example 17: If $2x^3 + ax^2 + bx + 4 = 0$ (a and b are positive real numbers) has 3 real roots, then prove that $a + b \geq 6\ (2^{1/3} + 4^{1/3})$.

Sol. Let α, β, γ be the roots of $2x^3 + ax^2 + bx + 4 = 0$. Given that all the coefficients are positive, so all the roots will be negative.

Let $\alpha = -\alpha$, $\alpha_2 = -\beta$, $\alpha_2 = -\gamma$

$\Rightarrow \alpha, \alpha_2\, \alpha_3 = \frac{a}{2}$

$\alpha_1\, \alpha_2\, \alpha_3 = 2$

Applying $AM \geq GM$, we have

$$\frac{\alpha_1 + \alpha_2 + \alpha_3}{3} \geq (\alpha_1\alpha_2\alpha_3)^{1/3} \Rightarrow a \geq 6 \times 2^{1/3}$$

$$\text{Also } \frac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3}{3} > (\alpha_1\alpha_2\alpha_3)^{2/3} \Rightarrow b \geq 6\ (4)^{1/3}$$

Therefore $a + b \geq 6(2^{1/3} + 4^{1/3})$.

Example 18: Prove that $a^4 + b^4 + c^4 \geq abc(a + b + c)$, $[a, b, c > 0]$

Sol. Using m^th power inequality, we get

$$\frac{a^4 + b^4 + c^4}{3} \geq \left(\frac{a+b+c}{3}\right)^4$$

$$= \left(\frac{a+b+c}{3}\right)\left(\frac{a+b+c}{3}\right)^3 \geq \left(\frac{a+b+c}{3}\right)[(abc)^{1/3}]^3$$

($\because$ A.M ≥ G.M

$$\text{or } \frac{a^4 + b^4 + c^4}{3} \geq \left(\frac{a+b+c}{3}\right)abc$$

$\therefore\ a^4 + b^4 + c^4 \geq abc(a + b + c)$.

Example 19: Prove that $\frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c} \geq \frac{9}{2}$, if $s = a + b + c$, $[a, b, c > 0]$

Sol. A.M. ≥ H.M.

$$\Rightarrow \frac{(a+b) + (b+c) + (c+a)}{3} \geq \frac{3}{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}}$$

$$\Rightarrow \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{9}{2(a+b+c)}$$

Concept Application

14. For what value of $n, \frac{a^{n+3}+b^{n+3}}{a^{n+2}+b^{n+2}}, a \neq b$ is the A.M. of a and b
15. If $x^6 - 12x^5 + ax^4 + bx^3 + cx^2 + dx + 64 = 0$ has positive roots then find a,b,c,d.
16. If a, b, c, d, e are positive real numbers, such that $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$, find the range of e.
17. If x, y, z are positive then prove that $(x + y)(y + z)(z + x)$

$$\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{z}+\frac{1}{x}\right) \geq 64.$$

ARITHMETICO-GEOMETRIC SERIES

A series each term of which is formed by multiplying the corresponding term of an AP & GP is called the Arithmetico-Geometric Series. e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5, ... are in AP & $1, x, x^2, x^3$ ..... are in GP.

Sum of n terms of an Arithmetico-Geometric Series

Let $S_n = a + (a + d)r + (a + 2d)r^2 + ... + [a + (n-1)d]\, r^{n-1}$.

then $S_n = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}, r \neq 1.$

Sum to Infinity: If $|r| < 1$ & $n \to \infty$ then $\lim\limits_{n \to \infty} r^n = 0$.

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}.$$

Train Your Brain

Example 20: Find the sum to n terms of the series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+.....$ Also find the sum if it exist if $n \to \infty$.

Sol. $S = \frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+.....+\frac{n}{2^n}$ ... (*i*)

$\frac{S}{2} = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+.....+\frac{n-1}{2^n}+\frac{n}{2^{n+1}}$ ... (*ii*)

By (*i*) – (*ii*)

$$\frac{S}{2} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+.....+\frac{1}{2^n}-\frac{n}{2^{n+1}}$$

$$= \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} - \frac{n}{2^{n+1}} = 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}}$$

$$S_n = 2\left[\frac{2^{n+1}-2-n}{2^{n+1}}\right] = \frac{2^{n+1}-2-n}{2^n}$$

If $n \to \infty$ then $S_\infty = \lim\limits_{n \to \infty}\left[2 - \frac{1}{2^{n-1}} - \frac{n}{2^n}\right] = 2$

Example 21: Find the sum of the series.

$$1 + 3.2 + 5.2^2 + 7.2^3 + \_ + (2n-1)\,2^{n-1}$$

Sol. $S_n = 1 + 3.2 + 5.2^2 + 7.2^3 + \_ + (2n-1)\,2^{n-1}$ ... (*i*)

$2S_n = 2 + 3.2^2 + 5.2^3 + \_ + (2n-3).2^{n-1} + (2n-1)2^n$ ... (*ii*)

Subtracting (*ii*) from (*i*) we get

$-S_n = 1 + 2.2 + 2.2^2 + 2.2^3 \_ + 2.2^{n-1} - (2n-1)\,2^n$

$$-S_n = 1 + \frac{4\left(2^{n-1}-1\right)}{2-1} - (2n-1)2^n$$

$S_n = (2n-1)\,2^n - 2^{n+1} + 3 = 2^{n+1}\,(n-1) - 2^n + 3$

Concept Application

18. Find the sum to n terms of the series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+......$ Also find the sum if it exist if n approaches ∞
19. If positive square root of, $a^{\frac{1}{a}} \cdot (2a)^{\frac{1}{2a}} \cdot (4a)^{\frac{1}{4a}} \cdot (8a)^{\frac{1}{8a}} ......\infty$. is $\frac{8}{27}$, find the value of 'a'.
20. Find sum to infinite terms of the series $1 + 2x + 3x^2 + 4x^3 + ..., -1 < x < 1$

SUMMATION

Sigma (Σ) Notation

Σ indicates sum i.e., $\sum\limits_{i=1}^{n} i = \sum n = 1 + 2 + 3 + \_\_ + n$

(*i*) $\sum\limits_{i=1}^{n} \frac{i+1}{i+2} = \frac{1+1}{1+2} + \frac{2+1}{2+2} + \frac{3+1}{3+2} + \_ + \frac{n+1}{n+2}$

(*ii*) $\sum\limits_{i=1}^{m} a = a + a + ... + a$ (m times) $= am$ where a is constant

(*iii*) $\sum\limits_{i=1}^{m} ai = a\sum\limits_{i=1}^{m} i = a(1 + 2 + \_ + m)$

(*iv*) $\sum\limits_{i=1}^{m} (i^3 - 2i^2 + i) = \sum\limits_{i=1}^{m} i^3 - 2\sum\limits_{i=1}^{m} i^2 + \sum\limits_{i=1}^{m} i$

Theorems

(*a*) $\sum\limits_{r=1}^{n} (a_r \pm b_r) = \sum\limits_{r=1}^{n} a_r \pm \sum\limits_{r=1}^{n} b_r$

(*b*) $\sum\limits_{r=1}^{n} ka_r = k\sum\limits_{r=1}^{n} a_r$

(*c*) $\sum\limits_{r=1}^{n} k = nk$ where k is a constant.

Important Results

(*i*) Sum of first n natural numbers

$$\Sigma n = 1+2+....+n = \frac{n(n+1)}{2}$$

(*ii*) Sum of squares of first n natural numbers

$$\Sigma n^2 = 1^2+2^2+.....+n^2 = \frac{n(n+1)(2n+1)}{6}$$

(*iii*) Sum of cubes of first n natural numbers

$$\Sigma n^3 = 1^3+2^3+3^3+.....+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = (\Sigma n)^2$$

(*iv*) Sum of n terms of a sequence $T_n = an^3+bn^2+cn+d$

$$\boxed{S_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + dn}$$

Train Your Brain

Example 22: Find the sum of the series 3.5 + 6.8 + 9.11 + ... + upto n terms

Sol. n^{th} term of 3, 6, 9, ... is $3n$

n^{th} term of 5, 8, 11, ... is $(3n+2)$

$\therefore\ T_n = 3n(3n+2) = 9n^2+6n$

$\therefore S_n = 9\Sigma n^2 + 6\Sigma n$

$$= \frac{9n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2}$$

$$= \frac{3}{2}n(n+1)[2n+1+2] = \frac{3n(n+1)(2n+3)}{2}$$

Example 23: Sum up to 16 terms of the series

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots \text{ is}$$

Sol. $t_n = \dfrac{1^3+2^3+3^3+\ldots+n^3}{1+3+5+....(2n-1)}$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\{2+2(n-1)\}} = \frac{\frac{n^2(n+1)^2}{4}}{n^2}$$

$$= \frac{(n+1)^2}{4} = \frac{n^2}{4}+\frac{n}{2}+\frac{1}{4} = \frac{1}{4}\Sigma n^2 + \frac{1}{2}\Sigma n + \frac{1}{4}\Sigma 1$$

$$= \frac{1}{4}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\cdot\frac{n(n+1)}{2} + \frac{1}{4}\cdot n$$

$$\therefore S_{16} = \frac{16.17.33}{24} + \frac{16.17}{4} + \frac{16}{4} = 446$$

Concept Application

21. Find the sum of n-terms of the series 2.5 + 5.8 + 8.11 +...
22. Evaluate $\sum\limits_{k=1}^{11}(2+3^k)$.

METHOD OF DIFFERENCES / TELESCOPIC METHOD / V_N METHOD

Let u_1, u_2, u_3...... be a sequence, such that $u_2-u_1, u_3-u_2, ...$ is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

$$S = u_1+u_2+u_3+.....+u_n \quad \ldots(i)$$

$$S = u_1+u_2+.....+u_{n-1}+u_n \quad \ldots(ii)$$

$(i)-(ii)$

$$\Rightarrow\ u_n = u_1+(u_2-u_1)+(u_3-u_2)+.....+(u_n-u_{n-1})$$

Where the series $(u_2-u_1)+(u_3-u_2)+.....+(u_n-u_{n-1})$ is either in A.P. or in G.P. then we can find u_n and hence sum of this series as $S = \sum\limits_{r=1}^{k} u_r$

Remark: It is not always necessary that the series of first order of differences i.e. $u_2-u_1, u_3-u_2 \ldots u_n-u_{n-1}$ is always either in A.P. or in G.P. in such case let $u_1 = T_1, u_2-u_1 = T_2, u_3-u_2 = T_3$ ....., $u_n-u_{n-1} = T_n$.

So $u_n = T_1+T_2+.....+T_n \quad \ldots(i)$

$u_n = T_1+T_2+\ldots+T_{n-1}+T_n \quad \ldots(ii)$

$(i)-(ii)$

$$\Rightarrow T_n = T_1+(T_2-T_1)+(T_3-T_2)+.....+(T_n+T_{n-1})$$

Now, the series $(T_2-T_1)+(T_3-T_2)+.....+(T_n-T_{n-1})$ is series of second order of differences and when it is either in A.P. or in G.P., then $u_n = u_1+\Sigma T_r$

Otherwise in the similar way we find series of higher order of differences and the n^{th} term of the series.

If possible express r^{th} term as difference of two terms as $t_r = f(r)-f(r\pm 1)$.

Train Your Brain

Example 24: Find the sum to n terms of the series, 0.7 + 7.7 + 0.77 + 77.7 + 0.777 + 777.7 + 0.7777 + ... where n is even.

Sol. $n = 2m$

$s = (0.7+0.77+0.777+\ldots\ m \text{ term})$
$+(7.7+77.7+777.7+\ldots m \text{ terms})$

$$= \frac{7}{9}(0.9+0.99+0.999+\ldots\ m \text{ terms})$$

$$+\frac{7}{90}((10^2-1)+(10^3-1)+\ldots+(10^{m+1}-1))$$

$$= \frac{7}{9}\left\{\frac{n}{2}-\frac{1}{9}\left(1-\frac{1}{10^{n/2}}\right)\right\} + \frac{7}{90}\left\{\frac{100}{9}\left(10^{n/2}-1\right)-\frac{n}{2}\right\}$$

Example 25: Find the sum to n-terms 3 + 7 + 13 + 21 + ...

Sol. Let $S = 3+7+13+21+.....+T_n \quad \ldots(i)$

$S = 3+7+13+.....+T_{n-1}+T_n \quad \ldots(ii)$

$(i)-(ii) \Rightarrow T_n = 3+4+6+8+.....+(T_n-T_{n-1})$

$$= 3+\frac{n-1}{2}[8+(n-2)2] = 3+(n-1)(n+2) = n^2+n+1$$

Hence $S = \Sigma(n^2+n+1) = \Sigma n^2+\Sigma n+\Sigma 1$

$$= \frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}+n = \frac{n}{3}(n^2+3n+5)$$

Example 26: Find the n^{th} term and the sum of n term of the series 2, 12, 36, 80, 150, 252

Sol. $S = 2+12+36+80+150+252+\ldots+T_n \quad \ldots(i)$

$S = 2+12+36+80+150+.....+T_{n-1}+T_n \quad \ldots(ii)$

$(i)-(ii)$

$\Rightarrow T_n = 2+10+24+44+70+102+\ldots+(T_n-T_{n-1}) \quad \ldots(iii)$

$T_n = 2+10+24+44+70+102$
$+\ldots+(T_{n-1}-T_{n-2})+(T_n-T_{n-1}) \quad \ldots(iv)$

$(iii)-(iv)$

$\Rightarrow T_n-T_{n-1} = 2+8+14+20+26+\ldots = \frac{n}{2}[4+(n-1)6]$

$= n[3n-1] \Rightarrow T_n-T_{n-1} = 3n^2-n$

$\therefore$ general term of given series is

$\Sigma T_n-T_{n-1} = \Sigma 3n^2-n = n^3+n^2.$

Hence sum of this sereis is

$$S = \Sigma n^3+\Sigma n^2 = \frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{12}(3n^2+7n+2) = \frac{1}{12}n(n+1)(n+2)(3n+1)$$

Example 27: Find the sum of the series to n terms whose general term is $2n+1$.

Sol. $S_n = \Sigma T_n = \Sigma(2n+1) = 2\Sigma n+\Sigma 1 = \frac{2(n+1)n}{2}+n = n^2+2n$.

Example 28:

(*a*) Sum the following series to infinity

$$\frac{1}{1\cdot4\cdot7}+\frac{1}{4\cdot7\cdot10}+\frac{1}{7\cdot10\cdot13}+.....$$

(*b*) Sum the following series upto n-terms

$1\cdot2\cdot3\cdot4+2\cdot3\cdot4\cdot5+3\cdot4\cdot5\cdot6+\ldots$

Sol. (*a*) $T_n = \dfrac{1}{[1+3(n-1)][1+3n][4+3n]}$

$$= \frac{1}{(3n-2)(3n+1)(3n+4)}$$

$$= \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)}-\frac{1}{(3n+1)(3n+4)}\right]$$

Credit:--- Divansh.....

$$T_1 = \frac{1}{6}\left[\frac{1}{1\cdot 4} - \frac{1}{4\cdot 7}\right]$$

$$T_2 = \frac{1}{6}\left[\frac{1}{4\cdot 7} - \frac{1}{7\cdot 10}\right]$$

$$T_n = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$\therefore S_n = \sum T_n = \frac{1}{6}\left[\frac{1}{1\cdot 4} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$= \left[\frac{1}{24} - \frac{1}{6(3n+1)(3n+4)}\right] \text{ as } n \to \infty$$

$$\therefore S_\infty = \frac{1}{24}$$

(*b*) $1\cdot 2\cdot 3\cdot 4 + 2\cdot 3\cdot 4\cdot 5 + 3\cdot 4\cdot 5\cdot 6 + \ldots\ldots$

$$T_n = \frac{1}{5} n(n+1)(n+2)(n+3)\,[(n+4)-(n-1)]$$

$$\therefore T_1 = \frac{1}{5}[1\cdot 2\cdot 3\cdot 4\cdot 5 - 0]$$

$$T_2 = \frac{1}{5}[2\cdot 3\cdot 4\cdot 5\cdot 6 - 1\cdot 2\cdot 3\cdot 4\cdot 5]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$T_n = \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4) - (n-1)n(n+1)(n+2)(n+3)]$$

$$\therefore S_n = \sum T_n$$

$$= \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4)]$$

$$= \frac{1}{5} n(n+1)(n+2)(n+3)(n+4)$$

Example 29: Sun to *n* terms of the series

$$\frac{4}{1.2.3} + \frac{5}{2.3.4} + \frac{6}{3.4.5} + \ldots\ldots$$

Sol. Let

$$T_r = \frac{r+3}{r(r+1)(r+2)} = \frac{1}{(r+1)(r+2)} + \frac{3}{r(r+1)(r+2)}$$

$$= \left[\frac{1}{r+1} - \frac{1}{r+2}\right] + \frac{3}{2}\left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right]$$

$$\therefore S = \left[\frac{1}{2} - \frac{1}{n+2}\right] + \frac{3}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$$

$$= \frac{5}{4} - \frac{1}{n+2}\left[1 + \frac{3}{2(n+1)}\right]$$

$$= \frac{5}{4} - \frac{1}{2(n+1)(n+2)}[2n+5]$$

Credit:--- Divansh.....

Example 30: Find the sum of the infinite series

$$\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\ldots..$$

Sol. Here $u_n=\dfrac{2n+1}{n^2(n+1)^2}=\dfrac{(n+1)^2-n^2}{n^2(n+1)^2}=\dfrac{1}{n^2}-\dfrac{1}{(n+1)^2}$

$$\therefore u_1=\frac{1}{1^2}-\frac{1}{2^2},$$

$$u_2=\frac{1}{2^2}-\frac{1}{3^2}\ldots..\,u_n=\frac{1}{n^2}-\frac{1}{(n+1)^2}$$

By addition, $S_n=1-\dfrac{1}{(n+1)^2}$

If $n\to\infty$, then $S_n\to 1$. Hence the sum of the infinite series is 1.

Example 31: If $1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\ldots=\log_e 2$, then sum

$$\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.7}+\frac{1}{4.9}+\ldots..$$

Sol. If S denote the sum of the given series, then

$$S=\frac{2}{2.3}+\frac{2}{4.5}+\frac{2}{6.7}+\frac{2}{8.9}+\ldots$$

$$=2\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{8}-\frac{1}{9}\right)+\ldots..\right]$$

$$=2\left[1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots\right)\right]$$

$$=2[1-\log_e 2]=2-2\log_e 2.$$

Example 32: If a, b, c, d be four distinct positive quantities in G.P., then show that

(a) $a+d>b+c$

(b) $\dfrac{1}{cd}+\dfrac{1}{ab}>2\left(\dfrac{1}{bd}+\dfrac{1}{ac}-\dfrac{1}{ad}\right)$

Sol. Since a, b, c, d are in G.P., therefore

(a) using A.M. > G.M., we have for the first three terms

$$\frac{a+c}{2}>b \text{ i.e. } a+c>2b \qquad \ldots(i)$$

$$\frac{b+d}{2}>c \text{ i.e. } b+d>2c \qquad \ldots(ii)$$

From results (i) and (ii), we have $a+c+b+d>2b+2c$ i.e. $a+d>b+c$ which is the desired result.

Credit:--- Divansh.....

(*b*) using G.M. > H.M., we have for the first three terms

$b > \frac{2ac}{a+c}$ i.e. $ab + bc > 2ac$ ... (*iii*)

and for the last three terms $c > \frac{2bd}{b+d}$

i.e. $bc + cd > 2bd$ ... (*iv*)

From results (*iii*) and (*iv*), we have

$ab + cd + 2bc > 2ac + 2bd$ i.e. $ab + cd > 2(ac + bd - bc)$

i.e. $\frac{1}{cd} + \frac{1}{ab} > 2\left(\frac{1}{bd} + \frac{1}{ac} - \frac{1}{ad}\right)$

[dividing both sides by abcd] which is the desired result.

Concept Application

23. If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$ then find $\sum_{r=1}^{n}\frac{1}{T_r}$.

24. The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9)..... and so on. Show that the sum of the numbers in n^{th} group is $n^3 + (n-1)^3$.

EXPONENTIAL SERIES

(*a*) *e* is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(*b*) Logarithms to the base '*e*' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm.**

(*c*) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\infty$; where *x* may be any real or complex number & $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(*d*) $a^x = 1 + \frac{x}{1!}\ell na + \frac{x^2}{2!}\ell n^2 a + \frac{x^3}{3!}\ell n^3 a + \ldots\infty$ where $a > 0$

(*e*) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\infty$

LOGARITHMIC SERIES

(*a*) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\infty$ where $-1 < x \le 1$

(*b*) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\infty$ where $-1 \le x < 1$

Remember:

(*i*) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\infty = \ln 2$ (*ii*) $e^{\ln x} = x$

(*iii*) $\ln 2 = 0.693$ (*iv*) $\ln 10 = 2.303$

Aarambh (Solved Examples)

1. If the sum of *n* terms of an A.P. is $(pn + qn^2)$, where *p* and *q* are constants, find the common difference.

Sol. Given $S_n = pn + qn^2$ ... (*i*)

Changing *n* to $n - 1$

$S_{n-1} = P(n-1) + q(n-1)^2$ ... (*ii*)

Substracting (*ii*) from (*i*), we obtain

$S_n - S_{n-1} = pn - p(n-1) + qn^2 - q(n-1)^2$

$\Rightarrow T_n = pn - pn + p + qn^2 - qn^2 - q + 2qn$

$\Rightarrow T_n = p - q + 2qn$ ... (*iii*)

Changing *n* to $n - 1$, we have

$T_{n-1} = p - q + 2q(n-1)$ ... (*iv*)

Subtracting (*iv*) from (*iii*), we get

$T_n - T_{n-1} = 2qn - 2q(n-1) = 2q$,

which is constant independent of *n*.

Hence, the series in reference is an A.P., with common difference = $2q$.

2. The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of the sides of the polygon.

Sol. Let the number of sides of the polygon be *n*, then the sum of all exterior angles = 360° and sum of all interior angles = $180°n - 360°$.

Also, smallest angle = 120° and the angles form the A. P. $120°, 125°, 130°, \ldots\ldots\ldots, 120° + (n-1)5°$

(There are *n* interior angles)

Since the sum of all the angles = $180°n - 360°$

$120° + 125° + 130° + \ldots\ldots$ upto *n* terms $= 180°(n-2)$

$\Rightarrow \{2 \times 120° + (n-1)5°\}n = 2 \times 180°(n-2)$

$\Rightarrow 240n + 5n(n-1) = 360n - 720$

$\Rightarrow 48n + n(n-1) = 72n - 72$

$\Rightarrow n^2 - 25n + 144 = 0$ (Dividing throughout by 5)

$n = \frac{25 \pm \sqrt{625 - 4\times1\times144}}{2\times1} = \frac{25\pm\sqrt{49}}{2} = \frac{25\pm7}{2} = 16, 9.$

If $n = 16$, then the greatest angle becomes $120° + (16-1)5 = 195° > 180°$, which is not possible.

Hence, $n = 9$.

($\because$ An interior angle of a polygon is always < 180°)

3. If the 4th, 10th and 16th terms of a GP. are *x*, *y* and *z* respectively. Prove that *x*, *y*, *z* are in GP.

Sol. Let the G.P. be a, ar, ar^2

4^{th} term $= ar^3 \Rightarrow x = ar^3$

then 10^{th} term $= ar^3 \Rightarrow x = ar^9$ ... (*ii*)

16^{th} term $= ar^{15} \Rightarrow z = ar^{15}$ ... (*iii*)

Multiplying (*i*) and (*iii*) we get

$xz = a^2r^3r^{15} \Rightarrow xz = a^2r^{18} = (ar^9)^2$

x, *y*, *z* are in G.P. (using (*iii*)

4. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter, find the amount spent on the postage when 8th set of letter is mailed.

Sol. Total number of letters posted = $4 + 4\times4 + 4\times4 + \ldots$ upto 8 terms = $4 + 4^2 + 4^3 + \ldots$ upto 8 terms

$= \frac{4\{4^8 - 1\}}{4-1} = \frac{4}{3}(65536 - 1) = 4 \times 21845 = 87380$

$\therefore$ Total amount spent on the postage = ₹ $\frac{1}{2}\times 87380$

= 87380 = ₹ 43690

5. Let *a*, *b* and *c* be three positive numbers then prove that

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9$

Sol. We know that for positive numbers AM > HM

$\Rightarrow \frac{a+b+c}{3} \ge \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9.$

6. Let $f(x) = x^5 + x^3 + 5x^2 + \frac{1}{x^7} + \frac{1}{x^8} + \frac{3}{x} + 1$ be a function defined for positive values of *x*, then find the least value of $f(x)$.

Sol. *x* is positive thats why we can think about AM ≥ GM

$f(x) = x^5 + x^3 + 5x^2 + \frac{1}{x^7} + \frac{1}{x^8} + \frac{3}{x} + 1$

$= x^5 + x^3 + x^2 + x^2 + x^2 + x^2 + x^2 + \frac{1}{x^7} + \frac{1}{x^8} + \frac{1}{x} + \frac{1}{x} + \frac{1}{x} + 1$

(split always in equal parts)

$\Rightarrow \frac{f(x)}{13} \ge (1)^{1/13}$ (using AM > GM)

$f(x) \ge 13$

7. Find the sum of the series

$1 + 3 + 7 + 13 + 21 + 31 + \ldots$ up to *n* terms

Sol. $S_n = 1 + 3 + 7 + 13 + 21 + 31 + \ldots + T_n$

$S_n = 1 + 3 + 7 + 13 + 21 \ldots + T_{n-1} + T_n$

$0 = 1 + (2 + 4 + 6 + \ldots \text{ up to } n-1) - T_n$

$\Rightarrow T_n = 1 + n(n-1) = n^2 - n + 1$

$$S_n = \Sigma T_n = \Sigma n^2 - \Sigma n + \Sigma 1$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n = \frac{n^3+2n}{3}$$

8. If $S_n = \sum_{r=1}^{n} \frac{r^2-2r-1}{2^{r+1}}$ then find the value of S_∞

Sol. $S_n = \sum_{r=1}^{n} \frac{r^2-2r-1}{2^{r+1}}$; $T_r = \frac{r^2-2r-1}{2^{r+1}}$

Try to split T_r in two parts such that

$$T_r = f(r) - f(r+1) \Rightarrow T_r = \frac{r^2}{2^r} - \frac{(r+1)^2}{2^{r+1}}$$

$$T_1 = \frac{1^2}{2} - \frac{2^2}{2^2},\ T_2 = \frac{2^2}{2^2} - \frac{3^2}{2^3},\ T_3 = \frac{3^2}{2^3} - \frac{4^2}{2^4}$$

$$T_n = \frac{n^2}{2^n} - \frac{(n+1)^2}{2^{n+1}}$$

$$S_n = T_1 + T_2 + \dots + T_n = \frac{1}{2} - \frac{(n+1)^2}{2^{n+1}}$$

$$S_\infty = \frac{1}{2}$$

9. Let, $A_1, A_2, \dots, A_m$ be non-empty subsets of $\{1, 2, 3, \dots, 100\}$ satisfying the following conditions:

(1) The numbers $|A_1|, |A_2|, \dots, |A_m|$ are distinct.

(2) $A_1, A_2, \dots, A_m$ are pairwise disjoint.

(Here, $|A|$ denotes the number of elements in the set A. Then the maximum possible value of m is

(*a*) 13 (*b*) 14 (*c*) 15 (*d*) 16

Sol. Each subset $A_1, A_2, \dots; A_m$ has distinct no. of elements and are pair-wise disjoint.

Therefore, for maximum value of 'm'

$|A_1| = 1, |A_2| = 2, \dots, |A_m| = m$

Also, $1 + 2 + 3 + \dots + m \le 100$

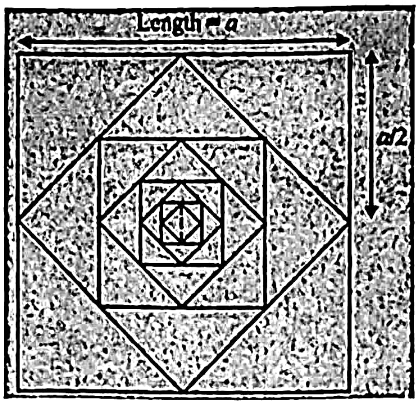
So, $m(m+1)/2 \le 100$

$m < 14$

Therefore, $m = 13$

Therefore, option (*a*) is the correct answer.

10. Let S_1 be the sum of areas of the squares whose sides are parallel to coordinate axes. Let S_2 be the sum of areas of the slanted squares as shown in the figure. Then S_1/S_2 is



(*a*) 2 (*b*) $\sqrt{2}$ (*c*) 1 (*d*) $1/\sqrt{2}$

Sol. $S_1 = a^2 + a^2/4 + a^2/16 + \dots \infty$

{Infinite G.P. First Term = a^2, Common ratio = 1/4}

$= a^2/(1 - 1/4) = 4a^2/3$

$S_2 = a^2/2 + a^2/8 + a^2/32 + \dots \infty$

{Infinite G.P. First Term = $a^2/2$, Common ratio = 1/4}

$= (a^2/2)/(1 - 1/4) = 2a^2/3$

Therefore, $S_1/S_2 = 2$

Therefore, option (*a*) is the correct answer.

11. Let A, G and H be the arithmetic mean. geometric mean and harmonic mean, respectively of two distinct positive real numbers. If α is the smallest of the two roots of the equation $A(G-H)x^2 + G(H-A)x + H(A-G) = 0$, then

(*a*) $-2 < \alpha < -1$ (*b*) $0 < \alpha < 1$

(*c*) $-1 < \alpha < 0$ (*d*) $1 < \alpha < 2$

Sol. Given, $A(G-H)x^2 + G(H-A)x + H(A-G) = 0$

Put $x = 1$ in above equation

$A(G-H) + G(H-A) + H(A-G) = 0$

$0 = 0$

$\Rightarrow$ $x = 1$ is one root (let it be 'α' and other root be 'β')

Now using product of roots,

$\alpha.\beta = A(A-G)/A(G-H)$

$1.\beta = (H.A - H.G)/A(G-H)$

$\beta = (G^2 - H.G)/A(G-H)\{A.H = G^2\}$

$\beta = G(G-H)/A(G-H)$

$\beta = G/A < 1$ {as $A > G$}

Therefore, option (*b*) is the correct answer.

School Level Problems

SINGLE CORRECT TYPE QUESTIONS

1. If the sum of n terms of an A.P. is given by $S_n = 3n + 2n^2$, then the common difference of the A.P. is

(*a*) 3 (*b*) 2 (*c*) 6 (*d*) 4

2. The lengths of three unequal edges of a rectangular solid block are in G.P. The volume of the block is 216 cm^3 and the total surface area is 252 cm^2. The length of the longest edge is

(*a*) 12 cm (*b*) 6 cm (*c*) 18 cm (*d*) 3 cm

3. If in an A.P., $S_n = qn^2$ and $S_m = qm^2$, where S_r denotes the sum of r terms of the A.P., then S_q equals

(*a*) $\frac{q^3}{2}$ (*b*) mnq

(*c*) q^3 (*d*) $(m+n)q^2$

4. If t_n denotes the nth term of the series $2 + 3 + 6 + 11 + 18 + \ldots$ then t_{50} is

(*a*) $49^2 - 1$ (*b*) 49^2

(*c*) $50^2 + 1$ (*d*) $49^2 + 2$

5. Let S_n denote the sum of the cubes of the first n natural numbers and Q_n denote the sum of the first n natural numbers. Then $\sum_{r=1}^{n} \frac{S_r}{Q_r}$ equals

(*a*) $\frac{n(n+1)(n+2)}{6}$ (*b*) $\frac{n(n+1)}{2}$

(*c*) $\frac{n^2+3n+2}{2}$ (*d*) None of these

6. Let S_{10} denote the sum of the first 10 terms of an A.P. If $S_{20} = 3S_{10}$ then $S_{30} : S_{10}$ is equal to

(*a*) 4 (*b*) 6 (*c*) 8 (*d*) 10

7. Sum of the series $\frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{4} + \ldots \infty$ is

(*a*) $1 + \frac{1}{\sqrt{2}}$ (*b*) 1

(*c*) $\frac{1}{\sqrt{2}-1}$ (*d*) $\frac{\sqrt{2}}{\sqrt{2}-1}$

8. If $\alpha^2, 2\beta^2, 3\gamma^2$ are in A.P., where the distinct numbers $\alpha^2, \beta^2, \gamma^2$ are in G.P. then the common ratio of the G.P. is

(*a*) 3 (*b*) $\frac{1}{3}$ (*c*) 2 (*d*) $\frac{1}{2}$

9. If $b + c, c + a, a + b$ are in H.P., then a^2, b^2, c^2 will be in

(*a*) G.P. (*b*) A.P.

(*c*) H.P. (*d*) none of these

VERY SHORT ANSWER TYPE QUESTIONS

10. Find the sum of series $1 + (1 + 2) + (1 + 2 + 3) + \ldots$ upto n terms.

11. If the n^{th} term of a series be $3 + n(n - 1)$, then find the sum of n terms of the series.

12. The first term of an A.P. is a, and the sum of the first p terms is zero, show that the sum of its next q terms is $\frac{-a(p+q)q}{p-1}$.

13. The digits of a three digit natural number are in AP and their sum is 15. The number obtained by reversing the digits is 396 less than the original number. Find the number.

14. Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

15. If a, b, c, d are in GP, then show that $a^2 + b^2, b^2 + c^2, c^2 + d^2$ are in GP

16. If the p^{th} and q^{th} terms of a G.P. are q and p respectively, show that its $(p + q)^{\text{th}}$ term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

LONG ANSWER TYPE QUESTIONS

17. If the sum of p terms of an A.P. is q and the sum of q terms is p, show that the sum of $p + q$ terms is $-(p + q)$. Also, find the sum of first $p - q$ terms $(p > q)$.

18. Find the sum of all the three digit numbers which leaves the reminder 2 when divided by 5.

19. A man accepts a position with an initial salary of ₹ 5200 per month. It is understood that he will receive an automatic increase of ₹ 320 in the very next month and each month thereafter.

(*a*) Find his salary for the tenth month.

(*b*) What is his total earnings during the first year?

20. Find the n^{th} term and the sum of n terms of the series

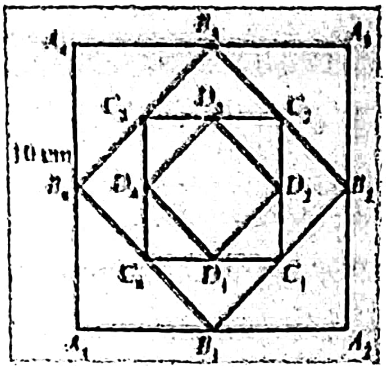
$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \ldots$$

21. If p^{th}, q^{th}, and r^{th} terms of an A.P. and G.P. are both a, b and c respectively, show that $a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$.

CASE STUDY BASED QUESTIONS

Case Study-I

A student of class XI draw a square of side 10 cm. Another student join the mid-point of this square to form new square. Again, the mid-points of the sides of this new square are joined to form another square by another student. This process is continued indefinitely.



Based on above information, answer the following questions.

22. The side of fourth square is (in cm)

(*a*) 5 (*b*) $\frac{\sqrt{5}}{2}$

(*c*) $\sqrt{5}$ (*d*) None of these

23. The perimeter of the 7th square is (in cm)

(*a*) 10 (*b*) 20 (*c*) 5 (*d*) $\frac{5}{2}$

Case Study-II

A sequence of non-zero numbers is said to be a geometric progression, if the ratio of each term, except the first one, by its preceding term is always constant. Rahul being a plant lover, decides to open a nursery and he bought few plants with pots. He wants to place pots in such a way that number of pots in first row is 2, in second row is 4 and in third row is 8 and so on....

Based on the above information, answer the following questions.

24. The constant multiple by which the number of pots is increasing in every row is

(*a*) 2 (*b*) 4 (*c*) 8 (*d*) 1

25. Total number of pots upto 10^{th} row is

(*a*) 1046 (*b*) 2046 (*c*) 1023 (*d*) 1024

Prarambh (Topicwise)

ARITHMETIC PROGRESSION AND ITS PROPERTIES

1. The sum of 24 terms of the following series $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+\ldots$ is

(*a*) 300 (*b*) $300\sqrt{2}$

(*c*) $200\sqrt{2}$ (*d*) $100\sqrt{2}$

2. If p^{th}, the q^{th} and r^{th} term of an arithmetic sequence are a, b and c respectively, then the value of $[a(q-r)+b(r-p)+c(p-q)]=$

(*a*) 1 (*b*) −1 (*c*) 0 (*d*) 1/2

3. Let T_r be the r^{th} term of an A.P. for r = 1, 2, 3, ... If for some positive integers m, n we have $T_m=\frac{1}{n}$ and $T_n=\frac{1}{m}$, then T_{mn} equal equals

(*a*) $\frac{1}{mn}$ (*b*) $\frac{1}{m}+\frac{1}{n}$ (*c*) 1 (*d*) 0

4. If 1, $\log_3(3^{1-x}+2)$, $\log_3(4.3^x-1)$ are in A.P. then x equals

(*a*) $\log_3 4$ (*b*) $1-\log_3 4$

(*c*) $1-\log_4 3$ (*d*) $\log_4 3$

5. The ratio of the sums of first n even numbers and odd numbers will be

(*a*) $1:n$ (*b*) $(n+1):1$

(*c*) $(n+1):n$ (*d*) $(n-1):1$

6. If S_n denotes the sum of n terms of an arithmetic progression, then the value of $(S_{2n}-S_n)$ is equal to

(*a*) $2S_n$ (*b*) S_{3n} (*c*) $\frac{1}{3}S_{3n}$ (*d*) $\frac{1}{2}S_n$

7. The sum of the integers from 1 to 100 which are not divisible by 3 or 5 is

(*a*) 2489 (*b*) 4735 (*c*) 2317 (*d*) 2632

8. Let S_n denotes the sum of n terms of an A.P.

If $S_{2n}=3S_n$, then ratio $\frac{S_{3n}}{S_n}=$

(*a*) 4 (*b*) 6 (*c*) 8 (*d*) 10

ARITHMETIC MEAN

9. If n A-M's are inserted between two quantities a and b, then their sum is equal to

(*a*) $n(a+b)$ (*b*) $\frac{n}{2}(a+b)$ (*c*) $2n(a+b)$ (*d*) $\frac{n}{2}(a-b)$

Credit:--- Divansh.....

10. If a, b, c, d, e, f are A.M.'s between 2 and 12, then $a + b + c + d + e + f$ is equal to

(a) 14 (b) 42

(c) 84 (d) None of these

GEOMETRIC PROGRESSION AND ITS PROPERTIES

11. If $x, 2x + 2, 3x + 3$, are in G.P., then the fourth term is

(a) 27 (b) -27 (c) 13.5 (d) -13.5

12. The 6^{th} term of a G.P. is 32 and its 8^{th} term is 128, then the common ratio of the G.P. is

(a) -1 (b) 2 (c) 4 (d) -4

13. The sum of 100 terms of the series $.9 + .09 + .009 \ldots$ will be

(a) $1-\left(\frac{1}{10}\right)^{100}$ (b) $1+\left(\frac{1}{10}\right)^{100}$

(c) $1-\left(\frac{1}{10}\right)^{106}$ (d) $1+\left(\frac{1}{10}\right)^{106}$

14. The sum of the series $6 + 66 + 666 + \ldots$ upto n terms is

(a) $(10^{n+1} - 9n + 10)/81$ (b) $2(10^{n+1} - 9n - 10)/27$

(c) $2(10^n - 9n - 10)/27$ (d) $(10^{n+1} - 9n - 10)/27$

15. Consider an infinite G.P. with first term a and common ratio r, its sum is 4 and the second term is 3/4, then

(a) $a = \frac{7}{4}, r = \frac{3}{7}$ (b) $a = \frac{3}{2}, r = \frac{1}{2}$

(c) $a = 2, r = \frac{3}{8}$ (d) $a = 3, r = \frac{1}{4}$

16. If are a, b, c are in G.P., then

(a) a^2, b^2, c^2 are in G.P.

(b) $a^2(b + c), c^2(a + b), b^2(a + c)$ are in G.P.

(c) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in G.P.

(d) a^2, b^2, c^2 are in A.P.

17. The third term of a G.P. is 4. The product of the first five terms is

(a) 4^3 (b) 4^5

(c) 4^4 (d) None of these

18. If sum of the infinite G.P., $p, 1, \frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}, \ldots$ is $\frac{9}{2}$, then value of p is

(a) $3, \frac{3}{2}$ (b) $\frac{2}{3}, 2$ (c) $\frac{3}{2}, 1$ (d) $\frac{1}{3}, 3$

GEOMETRIC MEAN

19. If G be the geometric mean of x and y, then $\frac{1}{G^2 - x^2} + \frac{1}{G^2 - y^2} =$

(a) G^2 (b) $\frac{1}{G^2}$

(c) $\frac{2}{G^2}$ (d) $3G^2$

20. The two geometric mean between the number 1 and 64 are

(a) 1 and 64 (b) 4 and 16

(c) 2 and 16 (d) 8 and 16

21. If a, b, c are in A.P., $b - a, c - b$ and a are in G.P., then $a : b : c$ is

(a) $1 : 2 : 3$ (b) $1 : 3 : 5$

(c) $2 : 3 : 4$ (d) $1 : 2 : 4$

A.G.P.

22. $1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + \ldots \infty$ is equal to

(a) 3 (b) 6 (c) 9 (d) 12

23. The sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \ldots$ is

(a) $2^n - n - 1$ (b) $1 - 2^{-n}$

(c) $n + 2^{-n} - 1$ (d) $2^n + 1$

24. The sum of the series $3 + 33 + 333 + \ldots + n$ terms is

(a) $\frac{1}{27}\left(10^{n+1} + 9n - 28\right)$ (b) $\frac{1}{27}\left(10^{n+1} - 9n - 10\right)$

(c) $\frac{1}{27}\left(10^{n+1} + 10n - 9\right)$ (d) None of these

25. The sum to n terms of the series $1 + 3 + 7 + 15 + 31 + \ldots$ is

(a) $2^{n+1} - n$ (b) $2^{n+1} - n - 2$

(c) $2^n - n - 2$ (d) None of these

HARMONIC PROGRESSION AND HARMONIC MEAN

26. Which number should be added to the numbers 13, 15, 19 so that the resulting numbers be the consecutive terms of H.P.

(a) 7 (b) 6 (c) -6 (d) -7

27. If 5^{th} term of a H.P. is $\frac{1}{45}$ and 11^{th} term is $\frac{1}{69}$, then its 16^{th} term will be

(a) 1/89 (b) 1/85

(c) 1/80 (d) 1/79

28. If the 7^{th} term of a H.P. is $\frac{1}{10}$ and the 12^{th} term is $\frac{1}{25}$, then the 20^{th} term is

(a) $\frac{1}{37}$ (b) $\frac{1}{41}$ (c) $\frac{1}{45}$ (d) $\frac{1}{49}$

Credit:--- Divansh.....

29. If H is the harmonic mean between p and q, then the value of $\frac{H}{p}+\frac{H}{q}$ is

(a) 2
(b) $\frac{pq}{p+q}$
(c) $\frac{p+q}{pq}$
(d) $\frac{2pq}{p+q}$

RELATION BETWEEN A.M., G.M., H.M.

30. If the arithmetic mean of two numbers be A and geometric mean be G, then the numbers will be

(a) $A\pm(A^2-G^2)$
(b) $\sqrt{A}\pm\sqrt{A^2-G^2}$
(c) $A\pm\sqrt{(A+G)(A-G)}$
(d) $\frac{A\pm\sqrt{(A+G)(A-G)}}{2}$

31. If the arithmetic, geometric and harmonic means between two positive real numbers be α, β and γ, then

(a) $\alpha^2=\beta\gamma$
(b) $\gamma^2=\alpha\beta$
(c) $\beta=\alpha\gamma$
(d) $\beta^2=\alpha\gamma$

32. If a^2, b^2, c^2 be in A.P., then $\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}$ will be in

(a) A.P.
(b) G.P.
(c) H.P.
(d) A.G.P.

33. If the A.M., G.M. and H.M. between two positive numbers a and b are equal, then

(a) $a=b$
(b) $ab=1$
(c) $a>b$
(d) $a<b$

34. If $\frac{a}{b},\frac{b}{c},\frac{c}{a}$ are in H.P., then

(a) a^2b, c^2a, b^2c are in A.P.
(b) a^2b, b^2c, c^2a are in H.P.
(c) a^2b, b^2c, c^2a are in G.P.
(d) None of these

35. If a, b, c are in G.P., then $\log_a x$, $\log_b x$, $\log_c x$ are in

(a) A.P.
(b) G.P.
(c) H.P.
(d) None of these

MISCELLANEOUS SERIES

36. $\frac{1^3+2^3+3^3+4^3+........12^3}{1^2+2^2+3^2+4^2+.........12^2}=$

(a) $\frac{234}{25}$
(b) $\frac{243}{35}$
(c) $\frac{263}{27}$
(d) $\frac{236}{35}$

37. If the sum of $1+\frac{1+2}{2}+\frac{1+2+3}{3}+.....$ to n terms is S, then S is equal to

(a) $\frac{n(n+3)}{4}$
(b) $\frac{n(n+2)}{4}$
(c) $\frac{n(n+1)(n+2)}{6}$
(d) n^2

38. The sum of the series $\frac{1}{3\times7}+\frac{1}{7\times11}+\frac{1}{11\times15}+....$ is

(a) $\frac{1}{3}$
(b) $\frac{1}{6}$
(c) $\frac{1}{9}$
(d) $\frac{1}{12}$

39. The first term of an A.P. of consecutive integer is p^2+1. The sum of $(2p+1)$ terms of this series can be expressed as

(a) $(p+1)^2$
(b) $(2p+1)(p+1)^2$
(c) $(p+1)^3$
(d) $p^3+(p+1)^3$

40. The sum of the series $\frac{1}{\log_2 4}+\frac{1}{\log_4 4}+\frac{1}{\log_8 4}+$

(a) $\frac{1}{2}n(n+1)$
(b) $\frac{1}{12}n(n+1)(2n+1)$
(c) $\frac{1}{n(n+1)}$
(d) $\frac{1}{4}n(n+1)$

41. Sum of the series $S=1^2-2^2+3^2-4^2+....-2002^2+2003^2$ is

(a) 2007006
(b) 1005004
(c) 2000506
(d) None of these

42. If a an b are p^{th} and q^{th} terms of an AP, then the sum of its $(p+q)$ terms is

(a) $\frac{p+q}{2}\left[a-b+\frac{a+b}{p-q}\right]$
(b) $\frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$
(c) $\frac{p-q}{2}\left[a+b+\frac{a+b}{p+q}\right]$
(d) None of these

43. If S is the sum of infinite terms of a G.P. whose first term is 'a', then the sum of the first n terms is

(a) $S\left(1-\frac{a}{S}\right)^n$
(b) $S\left[1-\left(1-\frac{a}{S}\right)^n\right]$
(c) $a\left[1-\left(1-\frac{a}{S}\right)^n\right]$
(d) None of these

44. If a, b, c are in A.P. and x, y, z are in G.P., then find the value of $x^{b-c}.y^{c-a}.z^{a-b}$.

(a) 0
(b) 1
(c) 4
(d) 2

45. If x, y, z are positive then the minimum value of $x^{\log y-\log z}+y^{\log z-\log x}+z^{\log x-\log y}$ is

(a) 3
(b) 1
(c) 9
(d) 16

Prabal (JEE Main Level)

1. If $a_1, a_2, a_3, \ldots$ are in A.P. such that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$, then $a_1 + a_2 + a_3 + \ldots + a_{23} + a_{24}$ is equal to
(*a*) 909 (*b*) 75 (*c*) 750 (*d*) 900

2. The first two terms of a geometric progression add up to 12. The sum of third and fourth terms is 48. If the terms are alternately positive and negative, then
(*a*) Common ratio of G.P. is –2
(*b*) Common ratio of G.P. is –3
(*c*) Sixth term of G.P. is 128
(*d*) None of these

3. The rational number, which equals the number $2.\overline{357}$ with recurring decimal is
(*a*) $\frac{2355}{1001}$ (*b*) $\frac{2379}{997}$
(*c*) $\frac{2355}{999}$ (*d*) None of these

4. Consider an A.P. with first term 'a' and the common difference 'd' Let S_k denote the sum of its first K terms. If $\frac{S_{kx}}{S_x}$ is independent of x, then
(*a*) $a = d/2$ (*b*) $a = d$
(*c*) $a = 2d$ (*d*) None of these

5. If $x \in R$, the numbers $5^{1+x} + 5^{1-x}$, $a/2$, $25^x + 25^{-x}$ form an A.P. then 'a' must lie in the interval:
(*a*) $[1, 5]$ (*b*) $[2, 5]$ (*c*) $[5, 12]$ (*d*) $[12, \infty)$

6. There are n A.M's between 3 and 54, such that the 8th mean : $(n-2)^{\text{th}}$ mean :: 3 : 5. The value of n is.
(*a*) 12 (*b*) 16 (*c*) 18 (*d*) 20

7. The sum to 10 terms of the series is $\sqrt{2} + \sqrt{6} + \sqrt{18} + \sqrt{54} + \ldots$ is
(*a*) $121\ (\sqrt{6} + \sqrt{2})$ (*b*) $\frac{121}{2}(\sqrt{3} + 1)$
(*c*) $243\ (\sqrt{3} + 1)$ (*d*) $243(\sqrt{3} - 1)$

8. For a sequence $\{a_n\}$, $a_1 = 2$ and $\frac{a_{n+1}}{a_n} = \frac{1}{3}$. Then $\sum_{r=1}^{20} a_r$ is
(*a*) $\frac{20}{2}[4 + 19 \times 3]$ (*b*) $3\left(1 - \frac{1}{3^{20}}\right)$
(*c*) $2(1 - 3^{20})$ (*d*) None of these

9. α, β be the roots of the equation $x^2 - 3x + a = 0$ and γ, δ the roots of $x^2 - 12x + b = 0$ and numbers α, β, γ, δ (in this order) form an increasing G.P., then
(*a*) $a = 3, b = 12$ (*b*) $a = 12, b = 3$
(*c*) $a = 2, b = 32$ (*d*) $a = 4, b = 16$

10. In a G.P. of positive terms, any term is equal to the sum of the next two terms. The common ratio of the G.P. is
(*a*) 2 cos 18° (*b*) sin 18°
(*c*) cos 18° (*d*) 2 sin 18°

11. If p is positive, then the sum to infinity of the series $\frac{1}{1+p} - \frac{1-p}{(1+p)^2} + \frac{(1-p)^2}{(1+p)^3} - \ldots$ is
(*a*) 1/2 (*b*) 3/4
(*c*) 1 (*d*) None of these

12. If $x > 0$, and $\log_2 x + \log_2 (\sqrt{x}) + \log_2 (\sqrt[4]{x}) + \log_2 (\sqrt[8]{x}) + \log_2 (\sqrt[16]{x}) + \ldots = 4$, then x equals
(*a*) 2 (*b*) 3 (*c*) 4 (*d*) 5

13. If $3 + \frac{1}{4}(3 + d) + \frac{1}{4^2}(3 + 2d) + \ldots$ upto $\infty = 8$, then the value of d is
(*a*) 9 (*b*) 5
(*c*) 1 (*d*) None of these

14. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in
(*a*) A.P. (*b*) G.P.
(*c*) H.P. (*d*) None of these

15. If $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$ where a, b, c are in AP and $|a| < 1, |b| < 1, |c| < 1$, then x, y, z are in
(*a*) HP
(*b*) Arithmetic-Geometric Progression
(*c*) AP
(*d*) GP

16. If a, b, c are in A.P. p, q, r are in H.P. and ap, bq, cr are in G.P., then $\frac{p}{r} + \frac{r}{p}$ is equal to
(*a*) $\frac{a}{c} + \frac{c}{a}$ (*b*) $\frac{a}{c} - \frac{c}{a}$ (*c*) $\frac{b}{q} + \frac{q}{b}$ (*d*) $\frac{b}{q} - \frac{c}{p}$

17. If $a^x = b^y = c^z = d^t$ and a, b, c, d are in G.P., then x, y, z, t are in
(*a*) A.P. (*b*) G.P.
(*c*) H.P. (*d*) None of these

18. If A, G & H are respectively the A.M., G.M. & H.M. of three positive numbers a, b, & c, then the equation whose roots are a, b & c is given by:
(*a*) $x^3 - 3Ax^2 + 3G^3x - G^3 = 0$
(*b*) $x^3 - 3Ax^2 + 3(G^3/H)x - G^3 = 0$
(*c*) $x^3 + 3Ax^2 + 3(G^3/H)x - G^3 = 0$
(*d*) $x^3 - 3Ax^2 - 3(G^3/H)x + G^3 = 0$

19. The sum $\sum_{r=2}^{\infty}\frac{1}{r^2-1}$ is equal to

(*a*) 1 (*b*) 3/4

(*c*) 4/3 (*d*) None of these

20. If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ upto $\infty = \frac{\pi^2}{6}$, then $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots =$

(*a*) $\pi^2/12$ (*b*) $\pi^2/24$

(*c*) $\pi^2/8$ (*d*) None of these

21. The sum of the first n-terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum is

(*a*) $\frac{n(n+1)^2}{4}$ (*b*) $\frac{n^2(n+2)}{4}$

(*c*) $\frac{n^2(n+1)}{2}$ (*d*) $\frac{n(n+2)^2}{4}$

22. $1^2 + 2^2 + \ldots + n^2 = 1015$, then value of n is

(*a*) 15 (*b*) 14

(*c*) 13 (*d*) None of these

$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)$ is equal to:

(*a*) -1 (*b*) 0

(*c*) 1 (*d*) None of these

24. If $[a_n]$ is an arithmetic sequence, then $\Delta = \begin{vmatrix} a_m & a_n & a_p \\ m & n & p \\ 1 & 1 & 1 \end{vmatrix}$ equal

(*a*) 1 (*b*) -1

(*c*) 0 (*d*) None of these

25. If the sum of first p terms, first q terms and first r terms of an A.P. be x, y and z respectively, then $\frac{x}{p}(q-r)+\frac{y}{q}(r-p)+\frac{z}{r}(p-q)$ is

(*a*) 0 (*b*) 2 (*c*) pqr (*d*) $\frac{8xyz}{pqr}$

26. If A_1, A_2; G_1, G_2 and H_1, H_2 be two A.M.'s, G.M.'s and H.M.'s between two quantities, then the value of $\frac{G_1G_2}{H_1H_2}$ is

(*a*) $\frac{A_1+A_2}{H_1+H_2}$ (*b*) $\frac{A_1-A_2}{H_1+H_2}$

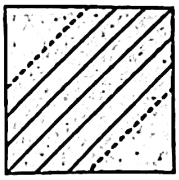
(*c*) $\frac{A_1+A_2}{H_1-H_2}$ (*d*) $\frac{A_1-A_2}{H_1-H_2}$

27. If the ratio of A.M. between two positive real numbers a and b to their H.M. is $m : n$, then $a : b$ is

(*a*) $\frac{\sqrt{m-n}+\sqrt{n}}{\sqrt{m-n}-\sqrt{n}}$ (*b*) $\frac{\sqrt{n}+\sqrt{m-n}}{\sqrt{n}-\sqrt{m-n}}$

(*c*) $\frac{\sqrt{m}+\sqrt{m-n}}{\sqrt{m}-\sqrt{m-n}}$ (*d*) None of these

28. In the given square, a diagonal is drawn and parallel line segments joining points on the adjacent sides are drawn on both sides of the diagonal. The length of the diagonal is $n\sqrt{2}\,cm$. If the distance between consecutive line segments be $\frac{1}{\sqrt{2}}cm$ then the sum of the lengths of all possible line segments and the diagonal is



(*a*) $n(n+1)\sqrt{2}\,cm$ (*b*) $n^2 cm$

(*c*) $n(n+2) cm$ (*d*) $n^2\sqrt{2}\,cm$

29. Greatest integer by which $1+\sum_{r=1}^{30} r \times r!$ is

(*a*) Composite number (*b*) Odd number

(*c*) Divisible by 3 (*d*) None of these

30. If $\sum_{r=1}^{n} r^4 = I(n)$ then $\sum_{r=1}^{n}(2r-1)^4$ is equal to

(*a*) $I(2n) - I(n)$ (*b*) $I(2n) - 16I(n)$

(*c*) $I(2n) - 8I(n)$ (*d*) $I(2n) - 4I(n)$

31. If $x_1, x_2, \ldots, x_{20}$ are in H.P. and $x_1, 2, x_{20}$ are in G.P., then $\sum_{r=1}^{19} x_r x_{r+1} =$

(*a*) 76 (*b*) 80

(*c*) 84 (*d*) None of these

32. The sum of the series $\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+\cdots$, to infinite terms, if $|x| < 1$, is

(*a*) $\frac{x}{1-x}$ (*b*) $\frac{1}{1-x}$ (*c*) $\frac{1+x}{1-x}$ (*d*) 1

33. If $1^2 + 2^2 + 3^2 + \ldots + 2003^2 = (2003)(4007)(334)$ and $(1)(2003) + (2)(2002) + (3)(2001) + \ldots + (2003)(1) = (2003)(334)(x)$, then x equals

(*a*) 2005 (*b*) 2004 (*c*) 2003 (*d*) 2001

34. If x, $|x + 1|$, $|x - 1|$ are first three terms of an A.P., then its sum upto 20 terms is:

(*a*) 180 (*b*) 350

(*c*) 90 (*d*) 720

35. Let $t_r = 2^{r/2} + 2^{-r/2}$. Then $\sum_{r=1}^{10} t_r^2$ is equal to

(*a*) $\frac{2^{21}-1}{2^{10}}+20$ (*b*) $\frac{2^{21}-1}{2^{10}}+19$

(*c*) $\frac{2^{21}-1}{2^{10}}-1$ (*d*) $\frac{2^{21}-1}{2^{10}}-19$

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36. If $a_1\, a_2\, a_3 \ldots a_n$ are in H. P. then $a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}\, a_n$ is equal to

(a) $(n-1)\, a_1a_n$ (b) na_1a_n

(c) $\frac{n}{2}(a_1 + a_n)$ (d) $(n+1)\, a_1a_n$

37. Sides of a triangle ABC; a, b, c are in $G.P.$ If 'r' be the common ratio of this of this $G.P.$, then

(a) $r \in \left(\frac{\sqrt{5}-1}{2}, \infty\right)$ (b) $r \in \left(\frac{\sqrt{5}+1}{2}, \infty\right)$

(c) $r \in \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)$ (d) $r \in \left(\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+3}{2}\right)$

38. If $a, b, c, d \in R^+$ and a, b, c, d are in $H.P.$, then

(a) $a+d > b+c$ (b) $a+b > c+d$

(c) $a+c > b+d$ (d) $a+b < c+d$

39. Let r_1, r_2, r_3, r_4 be the roots of the equation, $x^4 - 9x^3 + ax^2 + bx + 16 = 0$ where a, b are constants. Then the difference between the arithmetic mean $\frac{r_1+r_2+r_3+r_4}{4}$ and the geometric mean $\sqrt[4]{r_1r_2r_3r_4}$ of the roots, is equal to

(a) 5/2 (b) 1/2 (c) 17/4 (d) 1/4

40. If $a_1, a_2, a_3, \ldots\, a_{2n+1}$, are in A.P., then $\frac{a_{2n+1}-a_1}{a_{2n+1}+a_1} + \frac{a_{2n}-a_2}{a_{2n}+a_2} + \ldots + \frac{a_{n+2}-a_n}{a_{n+2}+a_n}$ is equal to

(a) $\frac{n(n+1)}{2} \frac{(a_2-a_1)}{a_{n+1}}$ (b) $\frac{n(n+1)}{2}$

(c) $\frac{(n-1)n(a_2-a_1)}{a_{n+1}}$ (d) $\frac{n(n-1)(n+1)(a_2-a_1)}{a_{n+1}}$

41. Let x_n, y_n, z_n, w_n denotes nth terms of four different arithmetic progressions with positive terms. If $x_4 + y_4 + z_4 + w_4 = 8$ and $x_{10} + y_{10} + z_{10} + w_{10} = 20$. then maximum value of $x_{20} \cdot y_{20} \cdot z_{20} \cdot w_{20}$ is

(a) 10^4 (b) 10^6 (c) 10^8 (d) 10^{10}

42. The sum of all positive integers n for which $\frac{1^3 + 2^3 + \ldots + (2n)^3}{1^2 + 2^2 + \ldots + n^2}$ is also an integer is.

(a) 8 (b) 9 (c) 15 (d) Infinite

43. Let x and y be two 2-digit numbers such that y is obtained by reversing the digits of x. Suppose they also satisfy $x^2 - y^2 = m^2$ for some positive integer. The value of $x + y + m$ is

(a) 88 (b) 112 (c) 144 (d) 154

44. The value of $\sum_{n=0}^{1947} \frac{1}{2^n + \sqrt{2^{1947}}}$ is equal to

(a) $\frac{487}{\sqrt{2^{1945}}}$ (b) $\frac{1946}{\sqrt{2^{1947}}}$

(c) $\frac{1947}{\sqrt{2^{1947}}}$ (d) $\frac{1943}{\sqrt{2^{1947}}}$

45. If n is the smallest natural number such that $n + 2n + 3n + \ldots + 99n$ is a perfect square, then the number of digits in n^2 is

(a) 1 (b) 2

(c) 3 (d) More than 3

46. Let a_n, $n \geq 1$, be an arithmetic progression with first term 2 and common difference 4, Let M_n be the average of the first n terms. Then the sum $\sum_{n=1}^{10} M_n$ is

(a) 110 (b) 335

(c) 770 (d) 1100

47. The number of three digit numbers $\overline{abc}$ such that the arithmetic mean of b and c and the square of their geometric mean are equal is:

(a) 9 (b) 18 (c) 36 (d) 54

48. If $a, b, c \in R^+$ then the minimum value of $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2)$ is equal to

(a) abc (b) $2abc$

(c) $3abc$ (d) $6abc$

49. If $t_r = \frac{r}{r^4+4}$ and $S_n = \sum_{r=1}^{n} t_r$ then the value of $37S_5 - \frac{7}{26}$ is equal to

(a) 13 (b) 25/26

(c) 1/2 (d) 7/26

50. If S denote the sum to infinity and S_n the sum of n terms of the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ such that $S - S_n < \frac{1}{300}$, then the least value of n is

(a) 4 (b) 5 (c) 6 (d) 7

51. If $a_1, a_2, a_3 \ldots$ are in A.P. and $b_k = a_k + a_{k-1} + \ldots + a_{k-n-1}$ $(k = 1, 2, 3, \ldots)$, then $b_1 + b_2 + \ldots + b_n$ is equal to

(a) $n(n+1)a_n$ (b) $(n-1)na_n$

(c) n^2a_n (d) $(n+1)^2a_n$

Parikshit (JEE Advanced Level)

SINGLE CORRECT TYPE QUESTIONS

1. The sum of 20 terms of the series whose r^{th} term is given by $T(n)=(-1)^n\dfrac{n^2+n+1}{n!}$ is

   (a) $\dfrac{20}{19!}-2$ (b) $\dfrac{21}{20!}-1$

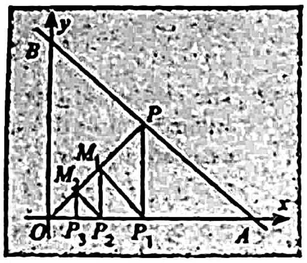
   (c) $\dfrac{21}{20!}$ (d) None of these

2. In a geometric series, the first term is a and common ratio is r.

   If S_n denotes the sum of the n terms and $U_n=\sum\limits_{n=1}^{n}S_n$ then $rS_n+(1-r)U_n$ equals

   (a) 0 (b) n (c) na (d) nar

3. The line $x+y=1$ meets x-axis at A and y-axis at B, P is the mid-point of AB;

   P is the foot of the perpendicular from P to OA;
   M_1 is that of P_1 from OP;
   P_2 is that of M_1 from OA;
   M_2 is that of P_2 from OP;
   P_3 is that of M_2. from OA;
   and so on.

   If P_n denotes the n^{th}' foot of the perpendicular on OA; then OP_n is

   (a) $\left(\dfrac{1}{2}\right)^{n-1}$ (b) $\left(\dfrac{1}{2}\right)^{n}$

   (c) $\left(\dfrac{1}{2}\right)^{n+1}$ (d) None of these

4. If x, y, and z are distinct prime numbers, then

   (a) x,y and z may be in A.P. but not in G.P.

   (b) x, y, and z may be in G.P. but not in A.P.

   (c) x, y, and z can neither be in A.P. nor in G.P.

   (d) None of these

5. Let x, y, z be positive reals. Which of the following implies $x=y=z$?

   I. $x^3+y^3+z^3=3xyz$

   II. $x^3+y^2z+yz^2=3xyz$

   III. $x^3+y^2z+z^2x=3xyz$

   IV. $(x+y+z)^3=27xyz$

   (a) I, IV only (b) I, II, IV only

   (c) I, II and III only (d) All of them

6. Let $a_1, a_2, \ldots\ldots\ldots\, a_{100}$ be non-zero real numbers such that $a_1+a_2+\ldots\ldots+a_{100}=0$, Then

   (a) $\sum\limits_{i=1}^{100}a_i2^{a_i}>0$ and $\sum\limits_{i=1}^{100}a_i2^{-a_i}<0$

   (b) $\sum\limits_{i=1}^{100}a_i2^{a_i}\geq 0$ and $\sum\limits_{i=1}^{100}a_i2^{-a_i}\geq 0$

   (c) $\sum\limits_{i=1}^{100}a_i2^{a_i}\leq 0$ and $\sum\limits_{i=1}^{100}a_i2^{-a_i}\geq 0$

   (d) $\sum\limits_{i=1}^{100}a_i2^{a_i}$ or $\sum\limits_{i=1}^{100}a_i2^{-a_i}$ depends on the choice of a_i's

7. Let $a_1, a_2, \ldots, a_n$ be the terms of a G.P. whose common ratio is r. set S_k denotes the sum of first k terms of the G.P. then the value of $\sum\limits_{1\leq i<j\leq m}a_ia_j$ in terms of S_{m-1} and S_m is

   (a) $rS_{m-1}S_m$ (b) $\dfrac{r}{1+r}S_{m-1}S_m$

   (c) $\dfrac{S_{m-1}S_m}{1+r}$ (d) $S_{m-1}S_m$

8. Let a, b, c be the sides of a triangle whose perimeter is P and area is A, then

   (a) $P^3\leq 27(b+c-a)(c+a-b)(a+b-c)$

   (b) $P^2\geq 3(a^2+b^2+c^2)$

   (c) $a^2+b^2+c^2\geq 4\sqrt{3}A$

   (d) $P^4<256A^2$

9. Consider a cuboid all of whose edges are integers and whose base is square. Suppose the sum of all its edges are numerically equal to the sum of the areas of all its six faces. Then the sum of all its edges is

   (a) 12 (b) 18 (c) 24 (d) 36

10. Let a and b be two positive real numbers such that $a+2b\leq 1$. Let A_1 and A_2 be, respectively, the areas of circles with radius ab^3 and b^2. Then the maximum possible value of A_1/A_2 is

    (a) 1/16 (b) 1/64

    (c) $1/16\sqrt{2}$ (d) 1/32

11. Largest term in the expansion of $\dfrac{1}{201}, \dfrac{4}{208}, \dfrac{9}{227}, \dfrac{16}{264}, \ldots$

    (a) 6th term (b) 7th term

    (c) 8th term (d) 10th term

12. Let each corner of a cube is represented by +1 or − 1 arbitrarily and on each of the six faces of the cube write the product of the numbers written at the four corners of that face. Let $S_1, S_2, \ldots, S_8$ are the numbers written on the corners and $P_1, P_2, \ldots, P_6$ are the product of the numbers written at

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the four corners of the face, then total number of solution of the equation $\sum_{i=1}^{8} S_i + \sum_{i=1}^{6} P_i = 0$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) infinite many

MULTIPLE CORRECT TYPE QUESTIONS

13. If positive numbers a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then

(*a*) $a = b = c$ (*b*) $2b = a + c$

(*c*) $b^2 = \sqrt{\frac{ac}{8}}$ (*d*) None of these

14. Let a, x, b be in A.P; a, y, b be in G.P. and a, z, b be in H.P. If $x = y + 2$ and $a = 5z$ then

(*a*) $y^2 = xz$

(*b*) $x > y > z$

(*c*) $a = 9, b = 1$

(*d*) $a = 1/4, b = 9/4$

15. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then

(*a*) $a + c = b + d$

(*b*) $e = 0$

(*c*) $a, b - 2/3, c - 1$ are in A.P.

(*d*) c/a is an integer

16. The value of $\sum_{r=1}^{n} \frac{1}{\sqrt{a+rx} + \sqrt{a+(r-1)x}}$ is

(*a*) $\frac{n}{\sqrt{a}+\sqrt{a+nx}}$ (*b*) $\frac{n}{\sqrt{a}-\sqrt{a+nx}}$

(*c*) $\frac{\sqrt{a+nx}-\sqrt{a}}{x}$ (*d*) $\frac{\sqrt{a}+\sqrt{a+nx}}{x}$

17. For the series, $S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \cdots$

(*a*) 7th term is 16

(*b*) 7th term is 18

(*c*) Sum of first 10 terms is $\frac{405}{4}$

(*d*) Sum of first 10 terms is $\frac{505}{4}$

18. The consecutive digits of a three digit number are in G.P. If the middle digit be increased by 2, then they form an A.P. If 792 is subtracted from this, then we get the number constituting of same three digits but in reverse order. Then number is divisible by

(*a*) 7 (*b*) 49

(*c*) 49 (*d*) None of these

19. Consider a sequence $\{a_n\}$ with $a_1 = 2$ and $a_n = \frac{a_{n-1}^2}{a_{n-2}}$ for all $n \geq 3$, terms of the sequence being distinct. Given that a_2 and a_5 are positive integers and $a_5 \leq 162$ then the possible value(s) of a_5 can be

(*a*) 162 (*b*) 64

(*c*) 32 (*d*) 2

20. A sequence $a_1, a_2, \ldots, a_n, \ldots$ of integers is defined by $a_{n+1} = a_n^2 - a_n + 1$ and $a_1 = 2$, then

(*a*) a_n is a prime number $\forall\, n \in N$

(*b*) a_n is an odd number $\forall\, n \in N, n \geq 2$

(*c*) HCF of $(a_i, a_j) = 1$ for $i \neq j$

(*d*) HCF of $(a_i, a_{i+2}) = 2\ \forall\, i \in N$

21. Let $f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) + \ldots + \frac{1}{2^{n-1}}\cos(a_n + x)$, where $a_1, a_2, \ldots, a_n$ are constant real numbers and $x \in R$. If $f(x_1) = f(x_2) = 0$ then $|x_2 - x_1|$ may be equal to

(*a*) π (*b*) 2π

(*c*) 3π (*d*) $\frac{\pi}{2}$

22. Let x, y, z are three real and distinct numbers which satisfy the equation $8(4x^2 + y^2) + 2z^2 - 4(4xy + yz + 2xz) = 0$, then

(*a*) $\frac{x}{y} = \frac{1}{2}$ (*b*) $\frac{y}{z} = \frac{1}{4}$

(*c*) $\frac{x}{y} = \frac{1}{3}$ (*d*) x, y, z are in G.P.

COMPREHENSION BASED QUESTIONS

Comprehension (Q. No. 23 to 25): Let $A_1, A_2, \ldots, A_m$ be the arithmetic means between 2 and 1027 and $G_1\ G_2\ G_3, \ldots\ G_n$ be the geometric means between 1 and 1024. The product of geometric means is 2^{45} and sum of arithmetic means is 1025 × 171.

23. The value of $\sum_{r=1}^{n} G_r$ is

(*a*) 512 (*b*) 2046

(*c*) 1022 (*d*) None of these

24. The number of arithmetic means is

(*a*) 442

(*b*) 342

(*c*) 378

(*d*) None of these

25. The numbers $2A_{171}, G_5^2 + 1, 2A_{172}$, are in

(*a*) A.P. (*b*) G.P.

(*c*) H.P. (*d*) None of these

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Comprehension (Q. No. 26 to 28): The numbers a, b, and c are between 2 and 18, such that

(i) their sum is 25

(ii) the numbers 2, a, and b are consecutive terms of an A.P.

(iii) the numbers b, c, 18 are consecutive terms of a G.P.

26. The value of abc is

(a) 300 (b) 450

(c) 720 (d) None of these

27. Roots of the equation $ax^2 + bx + c = 0$ are

(a) Real and positive (b) Real and negative

(c) Imaginary (d) Real ans of opposite sign

28. If a, b and c are roots of the equation $x^3 + qx^2 + rx + s = 0$, then the value of r is

(a) 184 (b) 196

(c) 224 (d) None of these

Comprehension (Q. No. 29 to 31): The sum of squares of three distinct real numbers which form an increasing GP is S^2 (common ratio is r). If sum of numbers is αS, then

29. If $r = 3$ then α^2 cannot lie in

(a) $\left(\frac{1}{3}, 1\right)$ (b) $(1, 2)$

(c) $\left(\frac{1}{3}, 3\right)$ (d) $(1, 3)$

30. If $\alpha^2 = 2$, then the value of common ratio r is greater than

(a) 9 (b) 4

(c) 2 (d) 3

31. If $r = 2$, then the value of α^2 is $\frac{a}{b}$ (where a and b are coprime) then $a + b$ is

(a) 3 (b) 5

(c) 8 (d) 10

Comprehension (Q. No. 32 to 34): Consider three distinct real numbers a, b, c in a G.P. with $a^2 + b^2 + c^2 = t^2$ and $a + b + c = \alpha t$. Sum of the common ratio and its reciprocal is denoted by S.

32. Complete set of α^2 is

(a) $\left(\frac{1}{3}, 3\right)$ (b) $\left[\frac{1}{3}, 3\right]$

(c) $\left(\frac{1}{3}, 3\right) - \{1\}$ (d) $\left(-\infty, \frac{1}{3}\right) \cup (3, \infty)$

33. Complete set of S is

(a) $(-2, 2)$ (b) $(-\infty, -2) \cup (2, \infty)$

(c) $(-1, 1)$ (d) $(-\infty, -1) \cup (1, \infty)$

34. If a, b, and c also represent the sides of a triangle, then the complete set of α^2 is

(a) $\left(\frac{1}{3}, 3\right)$ (b) $(2, 3)$

(c) $\left[\frac{1}{3}, 2\right]$ (d) $\left(\frac{\sqrt{5}+3}{2}, 3\right)$

Comprehension (Q. No. 35 to 36): If r^{th} term of a series can be written as $a_r = f(r) - f(r-1)$, then $S_n = \sum_{r=1}^{n} a_r = f(n) - f(0)$ and $S_\infty = \lim_{n\to\infty} S_n$, then

35. Value of $\sum_{r=1}^{\infty} \frac{(4r-1)5^r}{r^2+r}$ is

(a) 5 (b) -5

(c) 10 (d) none of these

36. Value of $(\cos 1 - 1)\sum_{n=1}^{50} \sin n + \sin 1 \sum_{n=1}^{50} \cos n$ is equal to

(a) $\sin 50^\circ$ (b) $\sin 50^\circ - \sin 1^\circ$

(c) $\sin 51^\circ - \sin 1^\circ$ (d) $\frac{1}{2}$

Comprehension (Q. No. 37 to 39): Let $M_n = \{0.a_1a_2a_3\ldots a_n \mid a_i = 0 \text{ or } 1\ \forall\ 1 \le i \le n-1 \text{ and } a_n = 1\}$ be a set of decimal fractions, T_n and S_n be the number of elements and sum of elements is M_n respectively then

37. $T_n =$

(a) 2^{n-1} (b) 2^n (c) 2^{n+1} (d) $2n - 1$

38. $S_n =$

(a) $\frac{2^{n-1}}{18}\left(-\frac{28}{10^n} + 1\right)$ (b) $\frac{2^{n-1}}{18}\left(1 + \frac{8}{10^n}\right)$

(c) $\frac{2^n}{18}\left(1 + \frac{18}{10^n}\right)$ (d) $\frac{2^n}{18}\left(-\frac{18}{10^n} + 1\right)$

39. $\lim_{n\to\infty} \frac{S_n}{T_n} =$

(a) $\frac{1}{10}$ (b) 18 (c) $-\frac{1}{10}$ (d) $\frac{1}{18}$

Comprehension (Q. No. 40 to 42): a, b, c, d are distinct positive numbers such that

Column-I		Column-II		Column-III	
I.	a, b, c, d are in A.P.	i.	$ab + cd > 2bc$	P.	$ac < bd$
II.	a, b, c, d are in G.P.	ii.	$bc > ad$	Q.	$ab + cd > 2ad$
III.	ab, bc, cd are in G.P.	iii.	$ab + bc < ad$	R.	$ab + bc + cd > 3ad$
IV.	ab, bc, cd are in H.P.	iv.	$a + d > b + c$	S.	$a + c < b + d$

Then, answers the following questions

40. Which of the following options is correct?

(a) (I) – (iii) – (Q) (b) (I) – (ii) – (R)

(c) (IV) – (iii) – (P) (d) (III) – (ii) – (R)

41. Which of the following options is correct?

(a) (II) - (iv) - (R) (b) (I) - (ii) - (P)

(c) (III) - (ii) - (R) (d) (IV) - (iii) - (Q)

Credit:--- Divansh.....

42. Which of the following options is correct?

(a) (I) - (ii) - (S) (b) (III) - (iv) - (P)

(c) (III) - (i) - (Q) (d) (III) - (ii) - (R)

MATCH THE COLUMN TYPE QUESTIONS

43. Match the column

	Column-I		Column-II
A.	If $\log_3 2$, $\log_3(2^x - 5)$ and $\log_3(2^x - 7/2)$ are in A.P., then value of x is equal to	p.	9
B.	Sum of infinite series $4 + \frac{8}{3} + \frac{12}{3^2} + \frac{16}{3^3} + \ldots$ is	q.	5
C.	The length, breadth, height of a rectangular box are in G.P. The volume is 27, the total surface area is 78. Then the length is	r.	3
D.	Between 1 and 31, m numbers have been inserted in such a way that the resulting sequence is an A.P. and the ratio of 7th and (m − 1)th numbers is 5 : 9. Find the value of m.	s.	4

(a) A → r, B → p, C → s, D → q

(b) A → r, B → p, C → s, D → p

(c) A → s, B → p, C → r, D → q

(d) A → p, B → r, C → s, D → q

44. Match the column

	Column -I		Column -I
A.	If a, b, c are in G.P., then $\log_a 10$, $\log_b 10$, $\log_c 10$ are in.	p.	A. P
B.	If $\frac{a+be^x}{a-be^x} = \frac{b+ce^x}{b-ce^x} = \frac{c+de^x}{c-de^x}$, then a, b, c, d are in	q.	H. P
C.	If a, b, c are in A.P.; a, x, b are in G.P. and b, y, c are in G.P., then x^2, b^2, y^2 are in	r.	G.P
D.	If x, y, z are in G.P., $a^x = b^y = c^z$ then $\log a, \log b, \log c$ are in	s.	None of these

(a) A → r, B → p, C → s, D → q

(b) A → q, B → r, C → p, D → r

(c) A → s, B → p, C → r, D → q

(d) A → p, B → r, C → s, D → q

45. Match the column

	Column -I		Column -I
A.	If $\sum r = 210$, then $\sum r^2$ is divisible by the greatest prime number which is greater than	p.	16
B.	Between 4 and 2916 is inserted odd number $(2n + 1)$ G.M's. Then the $(n + 1)$th G.M. is divisible by greatest odd integer which is less than	q.	10
C.	In a certain progression, four consecutive terms are 40, 30, 24, 20. Then the integral part of the next term of the progression is more than	r.	5
D.	$1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots\ldots = \frac{a}{b}$ where H.C.F.$(a, b) = 1$, then $a - b$ is less than	s.	20

(a) A → q, r, s; B → r, s; C → p, q; D → r, s

(b) A → p, q, r, s; B → r, s; C → p, q; D → r, s

(c) A → p, q, r, s; B → r, p; C → p, q; D → r, s

(d) A → p, q, r, s; B → r; C → p, q; D → r, s

INTEGER TYPE QUESTIONS

46. First term of an A.P. of non-constant terms is 3 and its second, tenth and thirty-fourth terms form a G.P., find the common difference.

47. If the roots of $10x^3 - nx^2 - 54x - 27 = 0$ are in harmonic progression, then 'n' equals.

48. The value of the $\sum_{n=1}^{\infty} \frac{2n+3}{3^n}$ is equal to.

49. The coefficient of the quadratic equation $ax^2 + (a + d)x + (a + 2d) = 0$ are consecutive terms of a positively valued, increasing arithmetic sequence. Then the least integral value of $\frac{d}{a}$ such that the equation has real solutions is

50. Consider an infinitely decreasing geometric progression b_1, b_2, b_3, ..., b_n, ... whose sum is 3 and sum of its cubes is $\frac{108}{13}$, find the value of $[b_2]$ (where $[\cdot]$ denotes greatest integer function.

51. If $\frac{1}{2.4} + \frac{1.3}{2.4.6} + \frac{1.3.5}{2.4.6.8} + \frac{1.3.5.7}{2.4.6.8.10} + \ldots \infty$ is equal to λ, then $[\lambda]$ = (where [ ] denote greatest integer function)

52. Given a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P. If $a = 2$ and $e = 18$, then the sum of all possible value of 'c' is.

53. Let sum of first three terms of G.P. with real terms is $\frac{13}{12}$ and 12 their product is −1. If the absolute value of the sum of their infinite terms is S, then find S

54. The p^{th}, q^{th} and r^{th} terms of a certain G.P. are R, S and T respectively. $R^{q-r} \cdot S^{r-p} \cdot T^{p-q} =$

Credit:--- Divansh.....

PYQ's (Past Year Questions)

ARITHMETIC PROGRESSION (AP)

1. In an A.P., the sixth term $a_6 = 2$. If the $a_1 a_4 a_5$ is the greatest, then the common difference of the A.P., is equal to **[29 Jan, 2024 (Shift-I)]**

(a) $\frac{3}{2}$ (b) $\frac{8}{5}$

(c) $\frac{2}{3}$ (d) $\frac{5}{8}$

2. If $\log_e a, \log_e b, \log_e c$ are in an A.P. and $\log_e a - \log_e 2b, \log_e 2b - \log_e 3c, \log_e 3c - \log_e a$ are also in an A.P, then $a : b : c$ is equal to **[29 Jan, 2024 (Shift-II)]**

(a) $9:6:4$ (b) $16:4:1$

(c) $25:10:4$ (d) $6:3:2$

3. The number of common terms in the progressions 4, 9, 14, 19, ..., up to 25th term and 3, 6, 9, 12, ..., up to 37th term is: **[27 Jan, 2024 (Shift-I)]**

(a) 9 (b) 5

(c) 7 (d) 8

4. The 20th term from the end of the progression

$20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots\ldots, -129\frac{1}{4}$ is: **[27 Jan, 2024 (Shift-II)]**

(a) -118 (b) -110

(c) -115 (d) -100

5. Let $a_1, a_2, a_3, \ldots$ be in an arithmetic progression of positive terms.

Let $A_k = a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2k-1}^2 - a_{2k}^2$

If $A_3 = -153, A_5 = -435$ and $a_1^2 + a_2^2 + a_3^2 = 66$, then $a_{17} - A_7$ is equal to ______. **[05 April, 2024 (Shift-I)]**

6. For $x \geq 0$, the least value of K, for which $4^{1+x} + 4^{1-x}$, $\frac{K}{2}, 16^x + 16^{-x}$ are three consecutive terms of an A.P. is equal to: **[05 April, 2024 (Shift-II)]**

(a) 10 (b) 4 (c) 8 (d) 16

7. A software company sets up m number of computer systems to finish an assignment in 17 days. If 4 computer systems crashed on the start of the second day, 4 more computer systems crashed on the start of the third day and so on, then it took 8 more days to finish the assignment. The value of m is equal to: **[06 April, 2024 (Shift-II)]**

(a) 125 (b) 150

(c) 180 (d) 160

8. An arithmetic progression is written in the following way

2

5 8

11 14 17

20 23 26 29

The sum of all the terms of the 10th row is ______. **[08 April, 2024 (Shift-II)]**

9. For three positive integers p, q, r, $x^{pq^2} = y^{qr} = z^{p^2 r}$ and $r = pq + 1$ such that $3, 3\log_y x, 3\log_z y, 7\log_x z$ are in A.P. with common difference $\frac{1}{2}$. Then $r - p - q$ is equal to **[24 Jan, 2023 (Shift-I)]**

(a) 2 (b) 6 (c) 12 (d) -6

10. If $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$, where a, b, c are in A.P. and $|a| < 1, |b| < 1, |c| < 1$, $abc \neq 0$, then **[27 June, 2022 (Shift-I)]**

(a) x, y, z are in A.P. (b) x, y, z are in G.P.

(c) $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P. (d) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 - (a+b+c)$

11. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is **[26 Feb, 2021 (Shift-II)]**

12. If the 10th term of an A.P. is $\frac{1}{20}$ and its 20th term is $\frac{1}{10}$, then the sum of its first 200 terms is **[08 Jan, 2020 (Shift-II)]**

(a) 100 (b) $50\frac{1}{4}$ (c) $100\frac{1}{2}$ (d) 50

13. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$2(a_1 + a_2 + \ldots + a_n) = b_1 + b_2 + \ldots + b_n$

holds for some positive integer n, is ______ **[JEE Adv, 2020]**

14. The sum of all two digit positive numbers which when divided by 7 yield 2 or 5 as remainder is: **[10 Jan, 2019 (Shift-I)]**

(a) 1256 (b) 1465 (c) 1365 (d) 1356

15. Let AP$(a; d)$ denote the set of all the term of an infinite arithmetic progression with first term a and common difference $d > 0$. If AP$(1; 3) \cap$ AP$(2; 5) \cap$ AP$(3; 7)$ = AP$(a; d)$ then $a + d$ equals ______ **[JEE Adv 2019]**

SUMMATION OF AP SERIES

16. Let S_n denote the sum of first n terms an arithmetic progression. If $S_{20} = 790$ and $S_{10} = 145$, then $S_{15} - S_5$ is: **[30 Jan, 2024 (Shift-I)]**

(a) 395 (b) 390
(c) 405 (d) 410

17. Let 2nd, 8th and 44th, terms of a non-constant A.P. be respectively the 1st, 2nd and 3rd terms of G.P. If the first term of A.P. is 1 then the sum of first 20 terms is equal to **[31 Jan, 2024 (Shift-II)]**

(a) 980 (b) 960
(c) 990 (d) 970

18. Let S_n be the sum to n-terms of an arithmetic progression 3, 7, 11, ...

If $40 < \left(\frac{6}{n(n+1)}\sum_{k=1}^{n} S_k\right) < 42$, then n equals ______.

[30 Jan, 2024 (Shift-II)]

19. Let 3, 7, 11, 15, ..., 403 and 2, 5, 8, 11, . . ., 404 be two arithmetic progressions. Then the sum, of the common terms in them, is equal to ________. **[01 Feb, 2024 (Shift-I)]**

20. Let S_n denote the sum of the first n terms of an arithmetic progression. If $S_{10} = 390$ and the ratio of the tenth and the fifth terms is 15 : 7, then $S_{15} - S_5$ is equal to: **[01 Feb, 2024 (Shift-II)]**

(a) 800 (b) 890 (c) 790 (d) 690

21. Let $a_1, a_2, a_3, ...$ be an A.P. If $a_7 = 3$, the product $a_1 a_4$ is minimum and the sum of its first n terms is zero, then $n! - 4a_{n(n+2)}$ is equal to: **[31 Jan, 2023 (Shift-II)]**

(a) 24 (b) $\frac{33}{4}$
(c) $\frac{381}{4}$ (d) 9

22. The series of positive multiples of 3 is divided into sets: {3}, {6, 9,12}, {15,18, 21, 24, 27},..... Then the sum of the elements in the 11th set is equal to.......... **[26 July, 2022 (Shift-I)]**

23. If sum of the first 21 terms of the series $\log_{9^{\frac{1}{2}}} x + \log_{9^{\frac{1}{2}}} x + \log_{9^{\frac{1}{4}}} x + ...$, where $x > 0$ is 504, then x is equal to: **[20 July 2021 (Shift-II)]**

(a) 7 (b) 9
(c) 243 (d) 81

24. Let $a_1, a_2, ... a_{30}$ be an A.P., $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=1}^{15} a_{2i-1}$. If $a_5 = 27$ and $S - 2T = 75$, then a_{10} is equal to **[09 Jan, 2019 (Shift-I)]**

(a) 52 (b) 57
(c) 47 (d) 42

GEOMETRIC PROGRESSION

25. Let a and b be be two distinct positive real numbers. Let 11th term of a GP, whose first term is a and third term is b, is equal to pth term of another GP, whose first term is a and fifth term is b. Then p is equal to **[30 Jan, 2024 (Shift-II)]**

(a) 20 (b) 25
(c) 21 (d) 24

26. If three successive terms of a G.P. with common ratio $r(r > 1)$ are the lengths of the sides of a triangle and $[r]$ denotes the greatest integer less than or equal to r, then $3[r] + [-r]$ is equal to: **[01 Feb, 2024 (Shift-II)]**

27. If the range of $f(\theta) = \frac{\sin^4\theta + 3\cos^2\theta}{\sin^4\theta + \cos^2\theta}, \theta \in R$ is $[\alpha, \beta]$, then the sum of the infinite G.P., whose first term is 64 and the common ratio is $\frac{\alpha}{\beta}$, is equal to **[08 April, 2024 (Shift-I)]**

28. In an increasing geometric progression of positive terms, the sum of the second and sixth terms is $\frac{70}{3}$ and the product of the third and fifth terms is 49. Then the sum of the 4th, 6th and 8th terms is: **[08 April, 2024 (Shift-II)]**

(a) 96 (b) 78 (c) 91 (d) 84

29. Let $a, ar, ar^2, ...,$ be an infinite G.P. If $\sum_{n=0}^{\infty} ar^n = 57$ and $\sum_{n=0}^{\infty} a^3 r^{3n} = 9747$, then $a + 18r$ is equal to: **[09 April, 2024 (Shift-II)]**

(a) 27 (b) 46 (c) 38 (d) 31

30. Let $a_1, a_2, a_3, ...$ be a G.P. of increasing positive numbers. Let the sum of its 6th and 8th terms be 2 and the product of its 3rd and 5th terms be $\frac{1}{9}$. Then $6(a_2 + a_4)(a_4 + a_6)$ is equal to **[13 April, 2023 (Shift-II)]**

(a) $2\sqrt{2}$ (b) 2 (c) $3\sqrt{3}$ (d) 3

31. The remainder on dividing $1 + 3 + 3^2 + 3^3 + \_\_ + 3^{2021}$ by 50 is.......... **[24 June, 2022 (Shift-II)]**

32. $\frac{1}{3^2-1} + \frac{1}{5^2-1} + \frac{1}{7^2-1} + ... + \frac{1}{(201)^2-1}$ is equal to **[24 Feb, 2021 (Shift-I)]**

(a) $\frac{101}{404}$ (b) $\frac{25}{101}$
(c) $\frac{101}{408}$ (d) $\frac{99}{400}$

33. If $x = \sum_{n=0}^{\infty}(-1)^n \tan^{2n}\theta$ and $y = \sum_{n=0}^{\infty}\cos^{2n}\theta$, for $0 < \theta < \frac{\pi}{4}$, then: **[09 Jan, 2020 (Shift-II)]**

(a) $y(1-x) = 1$ (b) $y(1+x) = 1$
(c) $x(1+y) = 1$ (d) $x(1-y) = 1$

Credit:--- Divansh.....

34. The product of three consecutive terms of a G.P. is [illegible] is added to each of the first and the second of these terms, the three terms now form an A.P. Then the sum of the original three terms of the given G.P. is: [12 Jan, 2019 (Shift-I)]

(*a*) 36 (*b*) 32
(*c*) 24 (*d*) 28

SUMMATION OF GP SERIES

35. If in a G.P. of 64 terms, the sum of all the terms is 7 times the sum of the odd terms of the G.P, then the common ratio of the G.P. is equal to [29 Jan, 2024 (Shift-I)]

(*a*) 7 (*b*) 4
(*c*) 5 (*d*) 6

36. If each term of a geometric progression $a_1, a_2, a_3, \ldots$ with $a_1 = \frac{1}{8}$ and $a_2 \neq a_1$, is the arithmetic mean of the next two terms and $S_n = a_1 + a_2 + \ldots + a_n$, then $S_{20} - S_{18}$ is equal to [29 Jan, 2024 (Shift-II)]

(*a*) 2^{15} (*b*) -2^{18}
(*c*) 2^{18} (*d*) -2^{15}

37. Let ABC be an equilateral triangle. A new triangle is formed by joining the middle points of all sides of the triangle ABC and the same process is repeated infinitely many times. If P is the sum of perimeters and Q is be the sum of areas of all the triangles formed in this process, then: [06 April, 2024 (Shift-II)]

(*a*) $P^2 = 36\sqrt{3}Q$ (*b*) $P^2 = 6\sqrt{3}Q$
(*c*) $P = 36\sqrt{3}Q^2$ (*d*) $P^2 = 72\sqrt{3}Q$

38. For $k \in N$, if the sum of the series $1 + \frac{4}{k} + \frac{8}{k^2} + \frac{13}{k^3} + \frac{19}{k^4} + \ldots$ is 10, then the value of k is [11 April, 2023 (Shift-II)]

39. If $A = \sum_{n=1}^{\infty} \frac{1}{\left(3+(-1)^n\right)^n}$ and $B = \sum_{n=1}^{\infty} \frac{(-1)^n}{\left(3+(-1)^n\right)^n}$, then $\frac{A}{B}$ is equal to: [26 June, 2022 (Shift-II)]

(*a*) $\frac{11}{9}$ (*b*) 1 (*c*) $-\frac{11}{9}$ (*d*) $-\frac{11}{3}$

40. In an increasing geometric series, the sum of the second and the sixth term is $\frac{25}{2}$ and the product of the third and fifth term is 25. Then, the sum of 4th, 6th and 8th terms is equal to: [26 Feb, 2021 (Shift-I)]

(*a*) 30 (*b*) 32 (*c*) 26 (*d*) 35

41. The product $2^{\frac{1}{4}}.4^{\frac{1}{16}}.8^{\frac{1}{48}}.16^{\frac{1}{128}} \ldots$ to ∞ is equal to [09 Jan, 2020 (Shift-I)]

(*a*) $2^{\frac{1}{4}}$ (*b*) $2^{\frac{1}{2}}$ (*c*) 1 (*d*) 2

AM-GM-HM, MISCELLANEOUS SEQUENCES

42. Let $\alpha = 1^2 + 4^2 + 8^2 + 13^2 + 19^2 + 26^2 + \ldots\ldots$ upto 10 terms and $\beta = \sum_{n=1}^{10} n^4$. If $4\alpha - \beta = 55k + 40$, then k is equal to______. [30 Jan, 2024 (Shift-I)]

43. The sum of the series $\frac{1}{1-3\cdot1^2+1^4} + \frac{2}{1-3\cdot2^2+2^4} + \frac{3}{1-3\cdot3^2+3^4} + \ldots$ up to 10 terms is [31 Jan, 2024 (Shift-I)]

(*a*) $\frac{45}{109}$ (*b*) $-\frac{45}{109}$ (*c*) $\frac{55}{109}$ (*d*) $-\frac{55}{109}$

44. If $8 = 3 + \frac{1}{4}(3+p) + \frac{1}{4^2}(3+2p) + \frac{1}{4^3}(3+3p) + \ldots\infty$, then the value of p is _____. [27 Jan, 2024 (Shift-I)]

45. Let 3, a, b, c be in A.P. and 3, $a-1$, $b+1$, $c+9$ be in G.P. Then, the arithmetic mean of a, b and c is: [01 Feb, 2024 (Shift-I)]

(*a*) -4 (*b*) -1 (*c*) 13 (*d*) 11

46. Let the first three terms 2, p and q with $q \neq 2$, of a G.P. be respectively the 7th, 8th and 13th terms of an A.P. If the 5th term of the G.P. is the nth term of the A.P., then n is equal to [04 April, 2024 (Shift-I)]

(*a*) 151 (*b*) 169
(*c*) 177 (*d*) 163

47. Let three real numbers a, b, c be in arithmetic progression and $a+1$, b, $c+3$ be in geometric progression. If $a > 10$ and the arithmetic mean of a, b and c is 8, then the cube of the geometric mean of a, b and c is [04 April, 2024 (Shift-II)]

(*a*) 120 (*b*) 312
(*c*) 316 (*d*) 128

48. The value of $\frac{1\times2^2+2\times3^2+\ldots+100\times(101)^2}{1^2\times2+2^2\times3+\ldots+100^2\times101}$ is [04 April, 2024 (Shift-II)]

(*a*) $\frac{306}{305}$ (*b*) $\frac{305}{301}$
(*c*) $\frac{32}{31}$ (*d*) $\frac{31}{30}$

49. If $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \ldots + \frac{1}{\sqrt{99}+\sqrt{100}} = m$ and $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \ldots + \frac{1}{99\cdot100} = n$, then the point (m, n) lies on the line [05 April, 2024 (Shift-I)]

(*a*) $11(x-1) - 100(y-2)\ 0$
(*b*) $11(x-2) - 100(y-1) = 0$
(*c*) $11(x-1) - 100y = 0$
(*d*) $11x - 100y = 0$

50. If $1 + \frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}} + \frac{5-2\sqrt{6}}{18} + \frac{9\sqrt{3}-11\sqrt{2}}{36\sqrt{3}} + \frac{49-20\sqrt{6}}{180} + \ldots$ upto $\infty = 2\left(\sqrt{\frac{b}{a}}+1\right)\log_e\left(\frac{a}{b}\right)$, where a and b are integers with $gcd(a, b) = 1$, then $11a + 18b$ is equal [05 April, 2024 (Shift-II)]

51. Let the first term of a series be $T_1 = 6$ and its r^{th} term $T_r = 3T_{r-1} + 6^r$, $r = 2, 3, \ldots\ldots, n$. If the sum of the first n terms of this series is $\frac{1}{5}(n^2 - 12n + 39)(4.6^n - 5.3^n + 1)$. Then n is equal to ________. [06 April, 2024 (Shift-I)]

52. If $S(x) = (1+x) + 2(1+x)^2 + 3(1+x)^3 + \ldots + 60(1+x)^{60}$, $x \neq 0$, and $(60)^2 S(60) = a(b)^b + b$, where $a, b \in N$, then $(a+b)$ equal to ________. [06 April, 2024 (Shift-II)]

53. Let the positive integers be written in the form:

1
2 3
4 5 6
7 8 9 10

If the k^{th} row contains exactly k numbers for every natural number k, then the row in which the number 5310 will be, is ______. [08 April, 2024 (Shift-I)]

54. If the sum of series $\frac{1}{1\cdot(1+d)} + \frac{1}{(1+d)(1+2d)} + \ldots\ldots\ldots + \frac{1}{(1+9d)(1+10d)}$ is equal to 5, then $50d$ is equal to: [09 April, 2024 (Shift-I)]

(*a*) 20 (*b*) 5
(*c*) 15 (*d*) 10

55. If $\left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2} + \ldots + \frac{1}{\alpha+1012}\right) - \left(\frac{1}{2\cdot1} + \frac{1}{4\cdot3} + \frac{1}{6\cdot5} + \ldots + \frac{1}{2024\cdot2023}\right) = \frac{1}{2024}$, then α is equal to: [09 April, 2024 (Shift-II)]

56. The sum $\sum_{n=1}^{\infty} \frac{2n^2+3n+4}{(2n)!}$ is equal to: [01 Feb, 2023 (Shift-II)]

(*a*) $\frac{11e}{2} + \frac{7}{2e}$ (*b*) $\frac{13e}{4} + \frac{5}{4e} - 4$
(*c*) $\frac{11e}{2} + \frac{7}{2e} - 4$ (*d*) $\frac{13e}{4} + \frac{5}{4e}$

57. Let $7\overset{r}{5\ldots5}7$ denote the $(r+2)$ digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S = 77 + 757 + 7557 + \ldots + 7\overset{98}{5\ldots5}7$. If $S = \dfrac{7\overset{99}{5\ldots5}7 + m}{n}$, where m and n are natural numbers less than 3000, then the value of $m + n$ is **[JEE Adv. 2023]**

58. Let $x, y > 0$. If $x^3y^2 = 2^{15}$, then the least value of $3x + 2y$ is **[24 June, 2022 (Shift-II)]**

(a) 30 (b) 32 (c) 36 (d) 40

59. The sum of the series $\dfrac{1}{x+1} + \dfrac{2}{x^2+1} + \dfrac{2^2}{x^4+1} + \ldots + \dfrac{2^{100}}{x^{2^{100}}+1}$ when $x = 2$ is: **[26 Aug, 2021 (Shift-I)]**

(a) $1 - \dfrac{2^{100}}{4^{100}-1}$ (b) $1 - \dfrac{2^{101}}{4^{101}-1}$

(c) $1 + \dfrac{2^{100}}{4^{101}-1}$ (d) $1 + \dfrac{2^{101}}{4^{101}-1}$

60. If $2^{10} + 2^9 \cdot 3^1 + 2^8 \cdot 3^2 + \ldots + 2 \cdot 3^9 + 3^{10} = S - 2^{11}$, then S is equal to: **[05 Sep, 2020 (Shift-I)]**

(a) $2 \cdot 3^{11}$ (b) $3^{11} - 2^{12}$ (c) $\dfrac{3^{11}}{2} + 2^{10}$ (d) 3^{11}

61. If $1 + (1 - 2^2 \cdot 1) + (1 - 4^2 \cdot 3) + (1 - 6^2 \cdot 5) + \ldots + (1 - 20^2 \cdot 19) = \alpha - 220\beta$, then an ordered pair (α, β) is equal to: **[04 Sep, 2020 (Shift-I)]**

(a) (10, 103) (b) (10, 97)

(c) (11, 97) (d) (11, 103)

62. Let $a_1, a_2, \ldots a_n$ be a given A.P. whose common difference is an integer and $S_n = a_1 + a_2 + \ldots + a_n$. If $a_1 = 1$, $a_n = 300$ and $15 \le n \le 50$, then the ordered pair (S_{n-4}, a_{n-4}) is equal to **[04 Sep, 2020 (Shift-II)]**

(a) (2490, 249)

(b) (2480, 249)

(c) (2490, 248)

(d) (480, 248)

63. The sum, $\sum_{n=1}^{7} \dfrac{n(n+1)(2n+1)}{4}$ is equal to **[08 Jan, 2020 (Shift-II)]**

64. The sum of the following series $1 + 6 + \dfrac{9(1^2+2^2+3^2)}{7} + \dfrac{12(1^2+2^2+3^2+4^2)}{9} + \dfrac{15(1^2+2^2+\ldots+5^2)}{11} + \ldots$ upto 15 terms, is: **[09 Jan, 2019 (Shift-II)]**

(a) 7820 (b) 7830

(c) 7520 (d) 7510

Credit:--- Divansh.....

PW Challengers

1. Let a and b be + ve integers such that

$$\frac{a}{b}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....-\frac{1}{1348}+\frac{1}{1349}.$$

then a is divisible by

(*a*) 2023 (*b*) 2024 (*c*) 2026 (*d*) NOT

2. Consider the sequence a_n defined by

$a_0=a_1=1$ and $a_2=1$ and $\begin{vmatrix} a_{n+3} & a_{n+2} \\ a_{n+1} & a_n \end{vmatrix}=n!$, if $a_{2025}=2^{\alpha}\times n!$

then, find the value of $\alpha-n$.

3. Consider the sequence a_n satisfying

$$a_{n+3}-a_{n+1}=\frac{1}{2}[a_{n+4}-a_{n+2}+a_n]$$

and $a_0=0$, $a_1=1$, $a_2=2$, $a_3=6$ then

(*a*) a_n is always divisible by $n!$

(*b*) a_n is always divisible by 2^n

(*c*) a_n is always divisible by n

(*d*) Ratio of a_n and F_n (Fibonacci sequence) is n

4. Consider the sequence U_n such that

$2U_{m+n}-U_{2n}=U_{2m}-U_{m-n}$, if $U_1=1$ then, find $\sum_{n=1}^{19}U_n-2025$

5. Consider the sequence U_n satisfying $U_1=1$

$16U_{n+1}=1+4U_n+\sqrt{1+24U_n}$ for $n\geq 1$

find $\left[\sum_{n=1}^{10}U_n\right]$ (where [.] is greatest integer function)

6. Consider the sequence ϕ_n such that, $\phi_0=1$, $\phi_1=3$, and $2\phi_{n+1}-\phi_n^2=1$, $n\geq 1$, then

find $\frac{1}{\phi_0+1}+\frac{1}{\phi_1+1}+...+\frac{1}{\phi_n+1}+\frac{1}{\phi_{n+1}-1}$

7. Let $f(x)=\frac{1}{\sqrt{x+\sqrt{x^2-1}}}$ and a sequence

$f(n)$ is defined on natural numbers.

If value of $\sum_{n=1}^{19601}f(n)=a+b\sqrt{2}$

then find value of $[a-b]$ (where [.] is G.I.F.).

8. If $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2m^2n}{3^m(n\cdot 3^m+m\cdot 3^n)}=\left(\frac{\alpha}{\beta}\right)^2$, then find $\alpha+\beta$

(where $gcd(\alpha,\beta)=1$, $\alpha,\beta\geq 0$).

9. If $\sum_{\substack{i=0\\ i\neq j\neq k}}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k}=\frac{\alpha}{\beta}$, ($\alpha$, β are co-prime positive integers) then find $3\alpha-\beta$.

10. If the sum of the series $1-\frac{1}{4}+\frac{1}{6}-\frac{1}{9}+\frac{1}{11}-\frac{1}{14}+\frac{1}{16}...\infty$ is $\frac{\pi}{\alpha}\left(\sqrt{\beta+2\sqrt{\gamma}}+\sqrt{\beta-2\sqrt{\gamma}}\right)$, then find $\alpha-\beta\gamma$.

> *"The pessimist sees difficulty in every opportunity. The optimist sees opportunity in every difficulty."*
>
> **– Winston Churchill**

ANSWER KEY

CONCEPT APPLICATION

1. 83667 2. $\frac{7}{16}$ 3. 293 5. 10 6. (*i*) 2, 4/2, 8/3, 16/4,....(*ii*) 2/3, 4/9, 2/27, 4/81..... 7. 2,6,18,54,162.....

8. $\frac{1}{3}$ 9. $2+\sqrt{3}$ 10. $\left(\frac{10(10^n-1)}{9}-n\right)$ 12. $\frac{1}{49}$ 13. $3ad$ 14. –2

15. $a=60, b=-160, c=240, d=-192$ 16. $\left[0,\frac{16}{5}\right]$ 18. 2 19. $\frac{1}{3}$ 20. $\frac{1}{(1-x)^2}$

21. $3n(n+1)^2-2n$ 22. $=\frac{41}{2}+\frac{3^{12}}{2}$ 23. $\frac{n^2+3n}{2(n+1)(n+2)}$

SCHOOL LEVEL PROBLEMS

1. (*d*) 2. (*a*) 3. (*c*) 4. (*d*) 5. (*a*) 6. (*b*) 7. (*a*) 8. (*b*) 9. (*b*) 22. (*d*)
23. (*c*) 24. (*a*) 25. (*b*)

PRARAMBH (TOPICWISE)

1. (*b*) 2. (*c*) 3. (*c*) 4. (*b*) 5. (*c*) 6. (*c*) 7. (*d*) 8. (*b*) 9. (*b*) 10. (*b*)
11. (*d*) 12. (*b*) 13. (*a*) 14. (*a*) 15. (*d*) 16. (*a*) 17. (*b*) 18. (*a*) 19. (*b*) 20. (*b*)
21. (*a*) 22. (*b*) 23. (*c*) 24. (*b*) 25. (*b*) 26. (*d*) 27. (*a*) 28. (*d*) 29. (*a*) 30. (*c*)
31. (*b*) 32. (*a*) 33. (*a*) 34. (*a*) 35. (*c*) 36. (*a*) 37. (*a*) 38. (*d*) 39. (*d*) 40. (*d*)
41. (*a*) 42. (*b*) 43. (*b*) 44. (*b*) 45. (*a*)

PRABAL (JEE MAIN LEVEL)

1. (*d*) 2. (*a*) 3. (*c*) 4. (*a*) 5. (*d*) 6. (*b*) 7. (*a*) 8. (*b*) 9. (*c*) 10. (*d*)
11. (*a*) 12. (*c*) 13. (*a*) 14. (*c*) 15. (*a*) 16. (*a*) 17. (*c*) 18. (*b*) 19. (*b*) 20. (*c*)
21. (*c*) 22. (*b*) 23. (*b*) 24. (*c*) 25. (*a*) 26. (*a*) 27. (*c*) 28. (*d*) 29. (*b*) 30. (*b*)
31. (*a*) 32. (*b*) 33. (*a*) 34. (*a*) 35. (*b*) 36. (*a*) 37. (*c*) 38. (*a*) 39. (*d*) 40. (*a*)
41. (*a*) 42. (*a*) 43. (*d*) 44. (*a*) 45. (*c*) 46. (*a*) 47. (*b*) 48. (*d*) 49. (*a*) 50. (*c*)
51. (*c*)

PARIKSHIT(JEE ADVANCED LEVEL)

1. (*b*) 2. (*c*) 3. (*b*) 4. (*a*) 5. (*b*) 6. (*a*) 7. (*b*) 8. (*c*) 9. (*c*) 10. (*b*)
11. (*b*) 12. (*a*) 13. (*a, b*) 14. (*a,b,c*) 15. (*a,b,c,d*) 16. (*a,c*) 17. (*a,d*) 18. (*a,b,c*) 19. (*a,c*) 20. (*b,c*)
21. (*a,b,c*) 22. (*a,d*) 23. (*c*) 24. (*b*) 25. (*a*) 26. (*d*) 27. (*c*) 28. (*b*) 29. (*a*) 30. (*c*)
31. (*d*) 32. (*c*) 33. (*b*) 34. (*d*) 35. (*d*) 36. (*c*) 37. (*a*) 38. (*b*) 39. (*d*) 40. (*b*)
41. (*a*) 42. (*c*) 43. (*b*) 44. (*b*) 45. (*b*) 46. [1] 47. [0] 48. [6] 49. [7] 50. [0]
51. [0] 52. [0] 53. [6] 54. [1]

Credit:--- Divansh.....

PYQ's (PAST YEAR QUESTIONS)

1. (*b*) **2.** (*a*) **3.** (*c*) **4.** (*c*) **5.** [910] **6.** (*a*) **7.** (*b*) **8.** [1505] **9.** (*a*) **10.** (*c*)
11. [1000] **12.** (*c*) **13.** [1] **14.** (*d*) **15.** [157] **16.** (*a*) **17.** (*d*) **18.** [9] **19.** [6699] **20.** (*c*)
21. (*a*) **22.** [6993] **23.** (*d*) **24.** (*a*) **25.** (*c*) **26.** [1] **27.** [96] **28.** (*c*) **29.** (*d*) **30.** (*d*)
31. [4] **32.** (*b*) **33.** (*a*) **34.** (*d*) **35.** (*d*) **36.** (*d*) **37.** (*a*) **38.** [2] **39.** (*c*) **40.** (*d*)
41. (*b*) **42.** [353] **43.** (*d*) **44.** [9] **45.** (*d*) **46.** (*d*) **47.** (*a*) **48.** (*b*) **49.** (*d*) **50.** [76]
51. [6] **52.** [3660] **53.** [103] **54.** (*b*) **55.** [1011] **56.** (*b*) **57.** [1219] **58.** (*d*) **59.** (*b*) **60.** (*d*)
61. (*d*) **62.** (*c*) **63.** [504] **64.** (*a*)

PW CHALLENGERS

1. (*b*) **2.** [0] **3.** (*c*,*d*) **4.** [445] **5.** [4] **6.** [1] **7.** [29] **8.** [7] **9.** [35] **10.** [0]

FOCUS ON IMPROVEMENT, NOT PERFECTION

- After each mock test, track your performance by identifying three areas for improvement.
- This approach helps maintain a growth mindset and encourages a positive outlook on learning opportunities.

Credit:--- Divansh.....

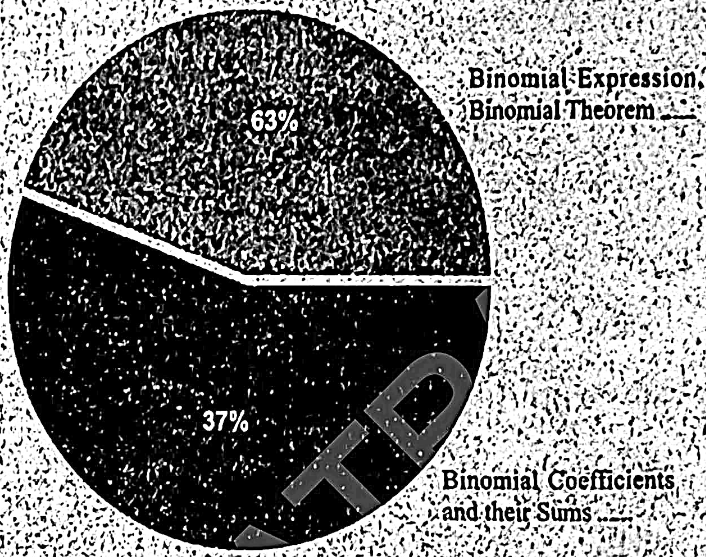
CHAPTER

7

Binomial Theorem

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
Pascal's Triangle

Topicwise Weightage of JEE Main 6 Years Paper (124 Sets)



"How's the Josh?" for these Topics: *Mark your confidence level in the blank space around the topic (Low-L, Medium-M, High-H)*

INTRODUCTION

Any algebraic expression consisting of only two different terms is known as binomial expression. The terms may consist of variables x, y etc. or constants or their mixed combinations.

Terminology used in Binomial Theorem

Factorial notation: $\underline{|n}$ or $n!$ is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3.2.1 & ; \text{ if } n \in N \\ 1 & ; \text{ if } n = 0 \end{cases}$$

Note: $n! = n \cdot (n-1)!; n \in N$

Mathematical meaning of ^nC_r: The term ^nC_r denotes number of combinations of r things choosen from n distinct things mathematically, $^nC_r = \dfrac{n!}{(n-r)!\,r!}, n \in N, r \in W, 0 \le r \le n$

Note: Other symbols of ^nC_r are $\begin{pmatrix} n \\ r \end{pmatrix}$ and $C(n, r)$.

Properties Related to ^nC_r

(*i*) $^nC_r = {^nC_{n-r}}$

Note: If $^nC_x = {^nC_y}$

$\Rightarrow$ Either $x = y$ or $x + y = n$

(*ii*) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(*iii*) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

(*iv*) $^nP_r = r!\,{^nC_r}$

(*v*) $^nC_r = \dfrac{n}{r}\,{^{n-1}C_{r-1}} = \dfrac{n(n-1)}{r(r-1)}\,{^{n-2}C_{r-2}} = ...$

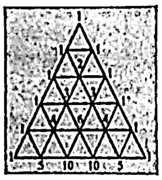
$$= \frac{n(n-1)(n-2).........(n-(r-1))}{r\,(r-1)(r-2).......2.1}$$

(*vi*) If n and r are relatively prime, then ^nC_r is divisible by n. But converse is not necessarily true.

Credit:--- Divansh.....

PASCAL'S TRIANGLE

Earlier people used to multiply the brackets to expand the given binomial of known index. Then came the Pascal triangle.



Note that

(a) The powers of x go down until it reaches zero, starting values is n.

(b) The power of y goes up from zero untill it reaches n.

(c) The n^{th} row of the Pascals triangle will be the coefficients of the expanded binomial of index n.

BINOMIAL THEOREM FOR POSITIVE INDEX

Binomial theorem gives a formula for the expansion of a binomial expression raised to any positive integral power. In general for a positive integer n

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 + \dots + {}^nC_nx^0y^n, \text{ where}$$

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

for $r = 0, 1, 2, \dots\dots, n$ is called binomial coefficient.

Observations

(a) The number of terms in the expansion is $(n + 1)$ i.e. one more than the index.

(b) The sum of the indices of x & y in each term is n.

(c) The binomial coefficients of the terms (${}^nC_0$, ${}^nC_1$...) equidistant from the beginning and the end are equal.

Proof of Binomial Theorem

The Binomial theorem can be proved by mathematical induction

Let $P(n)$ stands for the mathematical statement

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \dots + {}^nC_rx^{n-r}a^r + \dots + a^n \quad \dots (i)$$

Note: That there are $(n + 1)$ terms in R.H.S. and all the terms are of the same degree in x and a together.

When $n = 1$, L.H.S. $= x + a$ and R.H.S. $= x + a$ (there are only 2 terms)

$\therefore P(1)$ is verified to be true

Assume $P(m)$ to be true

$$\text{i.e., } (x+a)^m = x^m + {}^mC_1x^{m-1}a + {}^mC_2x^{m-2}a_2 + \dots + {}^mC_rx^{m-r}a^r + \dots + a^m \quad \dots (ii)$$

Multiplying equation (ii) by $(x + a)$, we have

$$(x+a)^m(x+a) = (x+a)\{x^m + {}^mC_1x^{m-1}a + {}^mC_2x^{m-2}a^2 + \dots + {}^mC_rx^{m-r}a^r + \dots + a^m\}$$

$$\text{i.e., } (x+a)^{m+1} = x^{m+1} + ({}^mC_1+1)x^ma + ({}^mC_2+{}^mC_1)x^{m-1}a^2 + \dots + ({}^mC_r+{}^mC_{r-1})x^{m-r+1}a^r + \dots + a^{m+1}$$

$$= x^{m+1} + {}^{(m+1)}C_1x^ma + {}^{(m+1)}C_2x^{m-1}a^2 + \dots + {}^{m+1}C_rx^{m+1-r}a^r + \dots a^{m+1} \quad \dots (iii)$$

(using the formula ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$)

Equation (iii) implies that $P(m + 1)$ is true and hence by induction $P(n)$ is true.

Alternative Method

By choosing x from all the brackets we get the term x^n. Choosing x from $(n - 1)$ factors and 'a' from the remaining factor we get $x^{n-1}a$. But the number of ways of doing this is equal to the number of ways of choosing one factor from n factors (i.e.,) ${}^nC_1$. Choosing x from $(n - 2)$ factor and a from the remaining two factors, we get $x^{n-2}a^2$. But the number of ways of doing this is equal to the number of ways of choosing two factors from n factors. i.e., ${}^nC_2$. Finally choosing 'a' from all the factors we get the term a^n.

$$\therefore (x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \dots + {}^nC_rx^{n-r}a^r + \dots + a^n$$

Train Your Brain

Example 1: The value of

$$\frac{(18^3+7^3+3.18.7.25)}{3^6+6.243.2+15.81.4+20.27.8+15.9.16+6.3.32+64} \text{ is}$$

Sol. The numerator is of the form

$a^3 + b^3 + 3ab(a + b) = (a + b)^3$ where $a = 18$ and $b = 7$

$\therefore Nr = (18 + 7)^3 = (25)^3$.

Denominator can be written as

$$3^6 + {}^6C_1.3^5.2^1 + {}^6C_2.3^4.2^2 + {}^6C_3.3^3.2^3 +$$

$${}^6C_43^2.2^4 + {}^6C_53.2^5 + {}^6C_62^6 = (3+2)^6 = 5^6 = (25)^3$$

$$\therefore \frac{N_r}{D_r} = \frac{(25)^3}{(25)^3} = 1$$

Example 2: Expand $\left(x-\frac{1}{x}\right)^6$

Sol. $\left(x-\frac{1}{x}\right)^6 = {}^6C_0x^6 + {}^6C_1x^5\left(\frac{-1}{x}\right) + {}^6C_2x^4\left(\frac{-1}{x}\right)^2$

$$+{}^6C_3x^3\left(\frac{-1}{x}\right)^3 + {}^6C_4x^2\left(\frac{-1}{x}\right)^4 + +{}^6C_5x\left(\frac{-1}{x}\right)^5 + {}^6C_6x^0\left(\frac{-1}{x}\right)^6$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

Concept Application

1. $(2n + 1)(2n + 3)(2n + 5)\dots(4n - 1)$ is equal to

(a) $\frac{(4n)!}{2^n \cdot (2n)!(2n)!}$ (b) $\frac{(4n)!n!}{2^n \cdot (2n)!(2n)!}$

(c) $\frac{(4n)!n!}{(2n)!(2n)!}$ (d) $\frac{(4n)!n!}{2^n!(2n)!}$

2. If $n \in N$ and is even, then

$$\frac{1}{1!(n-1)!} + \frac{1}{3!(n-3)!} + \frac{1}{5!(n-5)!} + \dots + \frac{1}{(n-1)!1!} =$$

(a) 2^n (b) $\frac{2^{n-1}}{n!}$ (c) $2^nn!$ (d) $\frac{2^{n+1}}{n!}$

GENERAL TERM IN THE BINOMIAL EXPANSION

The general term in the expansion of $(x + y)^n$ is $(r + 1)^{th}$ term. given by $T_{r+1} = {}^nC_rx^{n-r}y^r$ where $r = 0, 1, 2 \dots n$.

- Every term in the expansion is of n^th degree in variables x and y.
- The total number of terms in the expansion is $n + 1$.
- Binomial expansion can also be expressed as

  $$(x+y)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r.$$
- The binomial coefficients of the expansion equidistant from the beginning and the end are equal. In other words ${}^nC_r = {}^nC_{n-r}$.
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- **Term independent of x:** Term independent of x contains no x; Hence find the value of r for which the exponent of x is zero.
- **Term From ending:** Initial position of p^th term from end is $(n+2-p)^{th}$

Train Your Brain

Example 3: Find the 11^th term in the expansion of $\left(3x-\frac{1}{x\sqrt{3}}\right)^{20}$.

Sol. The general term

$$= T_{r+1} = (-1)^r\,{}^{20}C_r(3x)^{20-r}\left(\frac{1}{x\sqrt{3}}\right)^r$$

For the 11th term, we must take $r = 10$ ($\therefore r + 1 = 11$)

$$\therefore T_{11} = T_{10+1} = (-1)^{10}\,{}^{20}C_{10}(3x)^{20-10}\left(\frac{1}{x\sqrt{3}}\right)^{10}$$

$$= {}^{20}C_{10}3^{10}x^{10}\frac{1}{x^{10}\left(\sqrt{3}\right)^{10}} = {}^{20}C_{10}\left(\sqrt{3}\right)^{10} = {}^{20}C_{10}3^5$$

Example 4: Find the term independent of x in $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

Sol. The general term $= {}^9C_r\left(\frac{3x^2}{2}\right)^{9-r}\left(\frac{-1}{3x}\right)^r$

$$=(-1)^r\ {}^9C_r\frac{3^{9-2r}}{2^{9-r}}x^{18-3r}$$

The term independent of x, (or the constant term) corresponds to x^{18-3r} being x^0 or $18-3r=0 \Rightarrow r=6$

$\therefore$ The term independent of x is the 7th term and its value is

$$(-1)^6\ {}^9C_6\frac{3^{9-12}}{2^{9-6}}={}^9C_3\frac{3^{-3}}{2^3}=\frac{9.8.7}{3.2.1}\frac{1}{(6)^3}=\frac{7}{18}$$

Example 5: Find the coefficient of x^7 in the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$

Sol. In the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$, the general term is

$$T_{r+1}={}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r={}^{11}C_r.\frac{a^{11-r}}{b^r}x^{22-3r}$$

Putting $22-3r=7 \Rightarrow 3r=15 \Rightarrow r=5$

$$\therefore T_6={}^{11}C_5\frac{a^6}{b^5}.x^7$$

Hence, the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is ${}^{11}C_5\,a^6b^{-5}$.

Example 6: Find the number of rational terms in the expansion of $(9^{1/4}+8^{1/6})^{1000}$.

Sol. The general term in the expansion of $(9^{1/4}+8^{1/6})^{1000}$ is

$$T_{r+1}={}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r={}^{1000}C_r3^{\frac{1000-r}{2}}2^{\frac{r}{2}}$$

The above term will be rational if exponent of 3 and 2 are integers. i.e. $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is $\{0, 2, 4, \_\_\_\_\_ 1000\}$.
Hence, number of rational terms is 501

Concept Application

3. If the constant term of the binomial expansion $\left(2x-\frac{1}{x}\right)^n$ is -160, then n is equal to

(a) 4 (b) 6
(c) 8 (d) 10

Credit:--- Divansh.....

4. If the coefficients of x^4 and x^8 in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is:

(*a*) 15 (*b*) 45 (*c*) 55 (*d*) 56

5. Number of rational terms in the expansion of $(2^{1/7}+3^{1/4})^{198}$ is equal to

(*a*) 7 (*b*) 8 (*c*) 9 (*d*) 10

6. What is the sum of the coefficients of the polynomial after expansion of $(25-60x+22x^2+18x^3-6x^8)^{2013}$?

(*a*) 1 (*b*) −1 (*c*) 5262 (*d*) 0

7. Find the sum of following series:

(*i*) $\sum_{r=20}^{80} {}^{100-r}C_{20}$ (*ii*) $\sum_{r=10}^{100} {}^{r}C_{10}$

(*iii*) $\sum_{r=20}^{100} {}^{r}C_{10}$

8. Find the sum of following series $\sum_{r=0}^{n} {}^{n}C_r(2^r+3^r+4^r)$

MIDDLE TERMS OF THE

In the binomial expansion of $(x+y)^n$

When n is Odd

There are $(n+1)$ i.e. even terms in the expansion and hence two middle terms are given by

$$t_{\frac{n+1}{2}} = {}^{n}C_{\frac{n-1}{2}}\,x^{\frac{n+1}{2}}\,y^{\frac{n-1}{2}} \text{ for } r=\frac{n-1}{2}$$

and $$t_{\frac{n+3}{2}} = {}^{n}C_{\frac{n+1}{2}}\,x^{\frac{n-1}{2}}\,y^{\frac{n+1}{2}} \text{ for } r=\frac{n+1}{2}$$

When n is Even

There are odd terms in the expansion and hence, only one middle term is given by:

$$t_{\frac{n}{2}+1} = {}^{n}C_{n/2}\,x^{n/2}\,y^{n/2} \text{ for } r=\frac{n}{2}$$

Note: The greatest binomial coefficient in the expansion is always the binomial coefficient of middle term/terms.

Train Your Brain

Example 7: Find the middle term in the expression of $(1-2x+x^2)^n$.

Sol. $(1-2x+x^2)^n = [(1-x)^2]^n = (1-x)^{2n}$

Here, $2n$ is even integer, therefore, $\left(\frac{2n}{2}+1\right)^{\text{th}}$

i.e., $(n+1)^{\text{th}}$ term will be the middle term.

Now $(n+1)^{\text{th}}$ term in $(1-x)^{2n} = {}^{2n}C_n\,(1)^{2n-n}\,(-x)^n$

$$= {}^{2n}C_n(-x)^n = \frac{(2n)!}{n!\,n!}(-x)^n$$

Example 8: Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$

Sol. The number of terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e., $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+3}{2}\right)^{th}$ two middle terms. They are given by T_5 and T_6

$$\therefore T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4$$

$${}^9C_4 3^5 x^5 . \frac{x^{12}}{6^4} = \frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{3^5}{2^4 3^4}x^{17} = \frac{189}{8}x^{17}$$

$$\text{and } T_6 = T_{5+1} = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5 = -{}^9C_4 3^4 x^4 \frac{x^{15}}{6^5}$$

$$= \frac{-9.8.7.6}{1.2.3.4}\cdot\frac{3^4}{2^5.3^5}x^{19} = -\frac{21}{16}x^{19}$$

Example 9: The term independent of x in $\left[\sqrt{\left(\frac{3}{2x^2}\right)}+\sqrt{\frac{x}{3}}\right]^{10}$ is

Sol. General term in the expansion is

$${}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10}\cdot\frac{3^{5-r}}{2^{\frac{10-r}{2}}}.$$

For constant term, $\frac{3r}{2}=10$

$\Rightarrow r=\frac{20}{3}$ which is not an integer. Therefore, there will be no constant term.

Example 10: If the coefficient of $(2r+4)^{th}$ term and $(r-2)^{th}$ term in the expansion of $(1+x)^{18}$ are equal, find r.

Sol. Since coefficient of $(2r+4)^{th}$ term in $(1+x)^{18} = {}^{18}C_{2r+3}$.

Coefficient of $(r-2)^{th}$ term $= {}^{18}C_{r-3}$.

$\Rightarrow {}^{18}C_{2r+3} = {}^{18}C_{r-3} \Rightarrow 2r+3+r-3=18$

$\Rightarrow 3r=18 \Rightarrow r=6$

Concept Application

9. The coefficient of x^{49} in the expansion of $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)\dots\left(x-\frac{1}{2^{49}}\right)$ is equal to
   - (a) $-2\left(1-\frac{1}{2^{50}}\right)$
   - (b) Positive coefficient of x
   - (c) $2\left(1-\frac{1}{2^{49}}\right)$
   - (d) $-2\left(1-\frac{1}{2^{49}}\right)$

GREATEST TERM IN THE EXPANSION

The numerically greatest term (absolute value) in the expansion of $(1+x)^n$ is determined by the following process:

Let t_{r+1} be the numerically greatest term, then $|t_r| \le |t_{r+1}| \ge |t_{r+2}|$

Considering, $\left|\frac{t_{r+1}}{t_r}\right| \ge 1 \Rightarrow \left|\frac{n-r+1}{r}x\right| \ge 1$

$$\Rightarrow \frac{n-r+1}{r}|x| \ge 1 \Rightarrow (n-r+1)|x| \ge r$$

$$\Rightarrow r \le (n+1)\frac{|x|}{1+|x|} \quad \dots (i)$$

Also, $\left|\frac{t_{r+1}}{t_{r+2}}\right| \ge 1 \Rightarrow \left|\frac{r+1}{n-r}\frac{1}{x}\right| \ge 1 \Rightarrow r+1 \ge (n-r)|x|$

$$\Rightarrow r \ge \frac{n|x|-1}{1+|x|}$$

$$\Rightarrow r \ge (n+1)\frac{|x|}{1+|x|}-1 \quad \dots (ii)$$

Thus, from (i) and (ii) for t_{r+1} to be the greatest term

$r \le m$ and $r \ge m-1$ where, $m=(n+1)\frac{|x|}{1+|x|}$

For expansion of $(a+b)^n$

Case I: When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r=1, 2, 3 \dots, m-1)$

i.e., $T_2 > T_1, T_3 > T_2, \dots, T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when

$r=m$

i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when

$r > m$ $(r=m+1, m+2, \dots n)$

i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \dots T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude).

Case II: When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r < \frac{n+1}{1+\left|\frac{a}{b}\right|}$ $(r=1, 2, 3, \dots, m-1, m)$.

i.e. $T_2 > T_1, T_3 > T_2, \dots, T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r > \frac{n+1}{1+\left|\frac{a}{b}\right|}$ $(r=m+1, m+2, \dots n)$

i.e., $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \dots T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Note: (i) In any binomial expansion, the middle term(s) has **greatest binomial coefficient.**

In the expansion of $(a+b)^n$

If n	No. of greatest binomial coefficient	Greatest binomial coefficient
Even →	1	${}^nC_{n/2}$
Odd →	2	${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

(Values of both these coefficients are equal).

(ii) In order to obtain the term having **numerically greatest coefficient**, put $a=b=1$, and proceed as discussed above.

In general, for the expansion $(a+x)^n$ or $a^n\left(1+\frac{x}{a}\right)^n$ we can consider $m=(n+1)\frac{|x/a|}{1+|x/a|}$.

Train Your Brain

Example 11: Find the greatest term in the expansion of $(4+3x)^7$ when $x=\frac{2}{3}$.

Sol. Here the greatest term means the numerically greatest term.

$$\left|\frac{t_{r+1}}{t_r}\right| = \frac{{}^7C_r 4^{7-r}(3x)^r}{{}^7C_{r-1}4^{8-r}(3x)^{r-1}} = \frac{8-r}{r}\frac{3x}{4} = \frac{8-r}{2r} \text{ since } x=2/3$$

Now $|t_{r+1}| \ge |t_r|$ if $8-r \ge 2r$ or $\frac{8}{3} \ge r$

This inequality is valid only for $r=1$ or 2

Thus, for $r=1, 2$; $|t_{r+1}| > |t_r|$ and

for $r=3, 4$; $|t_{r+1}| < |t_r|$

$\therefore$ $|t_1| < |t_2| < |t_3| > |t_4| > |t_5| > \dots$

greatest term $= |t_3| = {}^7C_2 4^5 \cdot (3x)^2$ where $x=\frac{2}{3}$

$= 21 \times 4^5 \times 2^2 = 86016$

Concept Application

10. The greatest value of the terms independent of x in the expansion of $\left(x\sin\theta+\frac{\cos\theta}{x}\right)^{10}$ is

(a) ${}^{10}C_5$ (b) 2^5

(c) $2^5\cdot{}^{10}C_5$ (d) $\frac{{}^{10}C_5}{2^5}$

11. In the expansion of $(3^{x/4}+3^{5x/4})^n$ the sum of the binomial coefficients is 64 and the term with the greatest binomial coefficient exceeds the third term by $(n-1)$, then the value of x must be

(a) 1 (b) 2 (c) 0 (d) −1

12. Greatest term in the binomial expansion of $(a+2x)^9$ when $a=1$ and $x=\frac{1}{3}$ is

(a) 3rd and 4th (b) 4th and 5th

(c) only 4th (d) only 5th

13. Greatest term in the expansion of $(7-4x)^{10}$ is (when $x=1$)

(a) T_4 and T_5 (b) T_4 only

(c) T_5 only (d) T_5 and T_6

SUMMATION OF SERIES INCLUDING BINOMIAL COEFFICIENTS

In the binomial expansion of $(1+x)^n$, let us denote the coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_r, \ldots, {}^nC_n$ by $C_0, C_1, C_2, \ldots, C_r, \ldots, C_n$ respectively.

- The sum of the binomial coefficients in the expansion of $(1+x)^n$ is 2^n:

  $\because\ (1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

  Putting $x=1$,

  $\therefore\ 2^n = C_0 + C_1 + C_2 + \ldots + C_n$ ... (i)

  or $\sum_{r=0}^{n} C_r = 2^n$.

- The sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}.

  Since, $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$

  Putting $x=-1$,

  $0 = C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n$

  and $2^n = C_0 + C_1 + C_2 + C_3 + \ldots + C_n$ {from (i)}

  Adding and subtracting these two equations, we get

  $2^n = 2(C_0 + C_2 + C_4 + \ldots)$

  and $2^n = 2(C_1 + C_3 + C_5 + \ldots)$

  $\therefore C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

  sum of coefficients of odd terms = sum of coefficients of even terms = 2^{n-1}

Notes:

(a) $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$

(b) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

USE OF DIFFERENTIATION

It is used for BINO –AP sequences, if general term of AP is $T_r = a + rd$, then use the BINO expansion of $x^a(1+x^d)^n$ and differentiate both side.

Example: Find sum of $3\,{}^nC_0 - 8\,{}^nC_1 + 13\,{}^nC_2 - 18\,{}^nC_3 \ldots (n+1)$ terms.

Sol. General terms of 3, 8, 13, 18,

$T_r = 3 + r \times 5 = 3 + 5r$

Take expansion as

$x^3(1-x^5)^n = {}^nC_0x^3 - {}^nC_1x^8 + {}^3C_2x^{13} - {}^nC_3x^{18}\ldots$

Differentiate both side w.r.t x

$x^3 \times n(1-x^5)^{n-1} \times -5x^4 + (1-x^5)^n \times 3x^2$

$= 3\,{}^nC_0x^2 - 8\,{}^nC_1x^7 + 13\,{}^nC_2x^{12} - 18\,{}^nC_3x^{17}\ldots$

Put $x = 1 \Rightarrow 0 = 3\,{}^nC_0 - 8\,{}^nC_1 + 13\,{}^nC_2 - 18\,{}^nC_3\ldots$

USE OF INTEGRATION

We can use integration to find sum of given bino-hp series. expresssion of integration is decided by given sequence.

Example: Show that

$$2.C_0 + 2^2.\frac{C_1}{2} + 2^3.\frac{C_2}{3} + \ldots + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{(n+1)}-1}{n+1}$$

Sol. Integrating the expansion of $(1+x)^n$ between the limits 0 to 2.

$$\int_0^2 (1+x)^n\,dx = \int_0^2 (C_0 + C_1x + \ldots + C_nx^n)\,dx$$

$$\Rightarrow \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^2 = \left.C_0x + C_1\frac{x^2}{2} + \ldots + C_n\frac{x^{n+1}}{n+1}\right|_0^2$$

$$\Rightarrow 2.C_0 + 2^2.\frac{C_1}{2} + 2^3.\frac{C_2}{3} + \ldots. + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{(n+1)}-1}{n+1}$$

2nd Method: L.H.S.

$$= \frac{1}{n+1}\left\{{}^{(n+1)}C_1.2 + {}^{(n+1)}C_2.2^2 + {}^{(n+1)}C_3.2^3 + \ldots + {}^{(n+1)}C_{n+1}.2^{(n+1)}\right\}$$

$$= \frac{1}{n+1}\left\{1 + {}^{(n+1)}C_1.2 + {}^{(n+1)}C_2.2^2 + \ldots + {}^{(n+1)}C_{n+1}.2^{(n+1)} - 1\right\}$$

$$= \frac{1}{(n+1)}\left\{(1+2)^{n+1} - 1\right\} = \frac{3^{n+1}-1}{n+1}$$

VARIABLE UPPER INDEX

We can use the property of ${}^nC_r$ to evaluate the upper variable index. We can also use corresponding expansion to evaluate upper variable index.

Example: Find the sum of following series:

(i) $\sum_{r=1}^{51} \left({}^{(1001-r)}C_{950}\right)$ (ii) $\sum_{r=0}^{10} {}^{100+r}C_r$ (iii) $\sum_{r=0}^{20} {}^{m+n+r}C_r$

Credit:--- Divansh.....

Sol (i) $\sum_{r=1}^{51} {}^{1001-r}C_{950}$

$= {}^{1000}C_{950} + {}^{999}C_{950} + \dots\dots + {}^{951}C_{950} + {}^{950}C_{950}$

$= {}^{1000}C_{950} + {}^{999}C_{950} + \dots + {}^{951}C_{950} + {}^{951}C_{951}$

$= {}^{1000}C_{950} + {}^{999}C_{950} + \dots + {}^{952}C_{951}$

$= {}^{1000}C_{950} + {}^{1000}C_{951} = {}^{1001}C_{951}$

(ii) $\sum_{r=0}^{10} {}^{100+r}C_r = {}^{100}C_0 + {}^{101}C_1 + \dots + {}^{110}C_{10}$

$= {}^{101}C_0 + {}^{101}C_1 + \dots + {}^{110}C_{10}$

$= {}^{102}C_1 + \dots + {}^{110}C_{10} = {}^{110}C_{10} + {}^{110}C_9 = {}^{111}C_{10}.$

(iii) $\sum_{r=0}^{20} {}^{m+n+r}C_r$

$= {}^{m+n}C_0 + {}^{m+n+1}C_1 + {}^{m+n+2}C_2 + \dots + {}^{m+n+20}C_{20}$

$= {}^{m+n+1}C_0 + {}^{m+n+1}C_1 + {}^{m+n+2}C_2 + \dots\dots + {}^{m+n+20}C_{20}$

$= {}^{m+n+2}C_1 + {}^{m+n+2}C_2 + \dots + {}^{m+n+26}C_{20}$

$= {}^{m+n+20}C_{19} + {}^{m+n+20}C_{20} = {}^{m+n+21}C_{20}$

Bino-binomial Series

Product of two expansions can be used to solve some problems related to series of binomial coefficients in which each term is a product of two binomial coefficients.

(a) $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{n!n!}$

(b) $C_0 \cdot C_r + C_1 \cdot C_{r+1} + C_2 \cdot C_{r+2} + \dots + C_{n-r}$

$C_n = \dfrac{(2n)!}{(n+r)!(n-r)!} = {}^{2n}C_{n-r}$

Remember: $(2n)! = 2^n \cdot n!\,[1.3.5\dots(2n-1)]$

Train Your Brain

Example 12: Prove that the sum of the coefficients in the expansion of $(1 + x - 3x^2)^{2163}$ is -1.

Sol. Putting $x = 1$ in $(1 + x - 3x^2)^{2163}$, the required sum of coefficients $= (1 + 1 - 3)^{2163} = (-1)^{2163} = -1$.

Example 13: If the sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, then find the value of α.

Sol. Sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$ = Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$

Putting $x = y = 1$

$\therefore\ (\alpha - 1)^{35} = (1 - \alpha)^{35}$

$\Rightarrow\ (\alpha - 1)^{35} = -(\alpha - 1)^{35} \Rightarrow 2(\alpha - 1)^{35} = 0$

$\therefore\ \alpha - 1 = 0 \quad \therefore \alpha = 1$

Differentiation can be used to solve series in which each term is a product of an integer and a binomial coefficient i.e. in the form $k \cdot {}^nC_r$

Example 14: Show that $3 \cdot C_0 + 7 \cdot C_1 + 11 \cdot C_2 + \dots + (4n + 3)\, C_n = (2n + 3)\, 2^n$.

Sol. This problem can be done be differentiating the expansion of $x^3(1 + x^4)^n$ and putting $x = 1$.

$x^3(1 + x^4)^n = x^3(C_0 + C_1x^4 + C_2x^8 + \dots + C_nx^{4n})$

$= C_0x^3 + C_1x^7 + C_2x^{11} + \dots + C_nx^{4n+3}$

Differentiating we get.

$\Rightarrow\ 3x^2(1 + x^4)^n + x^3 n(1 + x^4)^{n-1} 4x^3$

$= 3x^2C_0 + 7x^6C_1 + 11x^{10}C_2 + \dots + (4n + 3)x^{4n+2}C_n$

Now substituting $x = 1$ in both sides.

$\Rightarrow\ 3C_0 + 7C_1 + 11C_2 + \dots + (4n + 3)C_n$

$= 3(2^n) + 4n(2)^{n-1} = (3 + 2n)2^n$

Example 15: Show that $C_1 - 2 \cdot C_2 + 3 \cdot C_3 - 4 \cdot C_4 + \dots + (-1)^{n-1}\, n \cdot C_n = 0$.

Sol. The problem can be done by differentiating the expansion of $(1 + x)^n$ and then putting $x = -1$.

2nd Method:

L.H.S. $= {}^nC_1 - 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 - \text{——} + (-1)^{n-1} \cdot n \cdot {}^nC_n$

$= n\{1 - {}^{(n-1)}C_1 + {}^{(n-1)}C_2 - {}^{n-1}C_3 + \text{—} + (-1)^{n-1}\, {}^{(n-1)}C_{n-1}\}$

$= n(1 - 1)^{n-1} = n \times 0 = 0$

Integration can be used to solve series in which each term is a binomial coefficient divided by an integer i.e. in the form $\dfrac{{}^nC_r}{k}$.

Example 16: If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ then prove that $\sum\sum_{0 \le i < j \le n} (C_i + C_j)^2 = (n-1)\,{}^{2n}C_n + 2^{2n}$

Sol. L.H.S $\sum\sum_{0 \le i < j \le n} (C_i + C_j)^2 = (C_0 + C_1)^2 + (C_0 + C_2)^2 + \dots$

$+ (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + \dots + (C_1 + C_n)^2$

$+ (C_2 + C_3)^2 + (C_2 + C_4)^2 + \dots + (C_2 + C_n)^2 + \dots + (C_{n-1} + C_n)^2$

$= n(C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2) + 2\sum\sum_{0 \le i < j \le n} C_iC_j$

$= n\,{}^{2n}C_n + 2\left\{2^{2n-1} - \dfrac{2n!}{2n!n!}\right\}$

$= n \cdot {}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n - 1) \cdot {}^{2n}C_n + 2^{2n} = \text{R.H.S.}$

Example 17: Prove that $\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1}=0$, where $k=(3n)/2$ and n is an even positive integer.

Sol. $(1+i\sqrt{3})^{3n}=\left({}^{3n}C_0-{}^{3n}C_2 3+{}^{3n}C_4 3^2-{}^{3n}C_6 3^3+\ldots\right)$

$$+i\sqrt{3}\left({}^{3n}C_1-{}^{3n}C_3 3+{}^{3n}C_5 3^2-{}^{3n}C_7 3^3+\ldots+(-3)^{\frac{3n}{2}-1}\,{}^{3n}C_{3n}\right.$$

$$\Rightarrow\ \sqrt{3}\left({}^{3n}C_1-{}^{3n}C_3 3+{}^{3n}C_5 3^2-{}^{3n}C_7 3^3+\ldots\right)$$

$$=I_m\left((1+i\sqrt{3})^{3n}\right)$$

$$=I_m\left(2e^{i\frac{\pi}{3}}\right)^{3n}=0$$

Example 18: If ${}^nJ_r=\dfrac{(1-x^n)(1-x^{n-1})\ldots(1-x^{n-r+1})}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^r)}$, prove that ${}^nJ_{n-r}={}^nJ_r$

Sol. $${}^nJ_r=\frac{(1-x^n)(1-x^{n-1})\ldots(1-x^{n-r+1})}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^r)}$$

$${}^nJ_{n-r}=\frac{(1-x^n)(1-x^{n-1})\ldots(1-x^{r+2})\ldots(1-x^{r+1})}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^{n-r})}$$

$$\frac{{}^nJ_{n-r}}{{}^nJ_r}=\frac{(1-x^n)(1-x^{n-1})\ldots(1-x^{r+1})}{(1-x)(1-x^2)\ldots(1-x^{n-r})}$$

$$=\frac{(1-x)(1-x^2)\ldots(1-x^r)}{(1-x^n)(1-x^{n-1})\ldots(1-x^{n-r+1})}$$

$$\frac{{}^nJ_{n-r}}{{}^nJ_r}=1 \qquad {}^nJ_{n-r}={}^nJ_r$$

Example 19: Find the sum of following series:

(i) $\sum_{r=0}^{49}{}^{100}C_r$ (ii) $\sum_{r=0}^{24}{}^{100}C_{4r+1}$

(iii) $\sum_{r=0}^{49}{}^{99}C_r$ (iv) $\sum_{r=0}^{n}{}^{2n+1}C_{2r+1}$

Sol. (i) $S=\sum_{r=0}^{100}{}^{100}c_r$

$$S={}^{100}C_0+{}^{100}C_1+\ldots+{}^{100}C_{49}$$

$$S={}^{100}C_{100}+{}^{100}C_{99}+{}^{100}C_{98}+\cdots+{}^{100}C_{51}$$

$$2S=2^{100}-{}^{100}C_{50}$$

$$S=\frac{2^{100}-{}^{100}C_{50}}{2}$$

(ii) $S=\sum_{r=0}^{24}{}^{100}C_{4r+1}$

$$S={}^{100}C_1+{}^{100}C_5+\ldots+{}^{100}C_{97}$$

$$(1+x)^{100}=\sum_{r=0}^{100}{}^nC_r x^r$$

$$x^3(1+x)^{100}=\sum_{r=0}^{100}{}^nC_r x^{r+3}$$

on putting $x=1$ ....

and adding above equation we get

$${}^{100}C_{4r+1}=2^{100}-(1+1)^{100}+0+(1-1)^{100}$$

$$=2^{100}+\left[(1-1)^{100}-(1+1)^{100}\right]$$

$$=2^{100}+\left[(-21)^{50}-(21)^{50}\right)$$

$$\sum_{r=0}^{24}{}^{100}c_{4r+1}=\frac{2^{100}}{4}=2^{98}$$

(iii) $\sum_{r=0}^{49}{}^{99}C_r$

$$S={}^{99}C_0+{}^{99}C_1+\ldots+{}^{99}C_{49}$$
$$S={}^{99}C_{99}+{}^{99}C_{98}+\ldots+{}^{99}C_{50}$$
$$2S=2^{99}$$
$$S=2^{98}$$

(iv) $\sum_{r=0}^{n}{}^{2n+1}C_{2r+1}$

$$S={}^{2n+1}c_1+{}^{2n+1}c_3+\cdots+{}^{2n+1}c_{2n+1}$$

$$S={}^{2n+1}C_{2n}+{}^{2n+1}C_{2n-2}+\cdots+{}^{2n+1}C_0$$

$$2S=2^{2n+1}$$

$$S=2^{2n}$$

Example 20: Find the sum of series:

$$\sum_{r=0}^{n}{}^nC_r(-1)^r\left(\frac{1}{2^r}+\frac{3^r}{3^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\ldots.\text{ upto } m \text{ terms}\right)$$

Sol. $\sum_{r=0}^{n}(-1)^n\cdot{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\ldots\right]$ upto m terms

$$=\sum_{r=0}^{n}(-1)^n\,{}^nC_r\left(\frac{1}{2}\right)^r+\sum_{r=0}^{n}(-1)^n\,{}^nC_r\left(\frac{3}{4}\right)^r+$$

$$\sum_{r=0}^{100}(-1)^{n}\,{}^nC_r\left(\frac{7}{8}\right)^r+\ldots$$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+$$

$$\ldots\left[\because\sum_{r=0}^{n}(-1)^n\,{}^nC_r x^r=(1-x)^r\right]$$

$$=\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n+\left(\frac{1}{8}\right)^n+\ldots=\left(\frac{1}{2}\right)^n\left[\frac{1-\left(\frac{1}{2^n}\right)^m}{1-\left(\frac{1}{2}\right)^n}\right]$$

$$=\frac{2^{mn}-1}{2^{mn}\cdot\left(2^n-1\right)}$$

Example 21: Prove that:

$$\sqrt{C_1}+\sqrt{C_2}+\sqrt{C_3}+\ldots+\sqrt{C_n}\le 2^{n-1}+\frac{n-1}{2}$$

Sol. $\sqrt{C_1}+\sqrt{C_2}+\sqrt{C_3}+\ldots+\sqrt{C_n}\le\frac{2^n+(n-1)}{2}$

$2\sqrt{C_1}+2\sqrt{C_2}+\ldots+2\sqrt{C_n}\le C_o+C_1+\ldots+C_n$

$+1+1+\ldots+1$ $(n-1)$ times

$C_o+\left(\sqrt{C_1}-1\right)^2+\left(\sqrt{C_2}-1\right)^2+\ldots+\left(\sqrt{C_2}-1\right)^2\ge 0$

Always true.

Concept Application

14. The sum of the series

$$\frac{{}^{101}C_1}{{}^{101}C_0}+\frac{2\cdot{}^{101}C_2}{{}^{101}C_1}+\frac{3\cdot{}^{101}C_3}{{}^{101}C_2}+\ldots+\frac{101\cdot{}^{101}C_{101}}{{}^{101}C_{100}}$$ equals

(*a*) 5050 (*b*) 4950 (*c*) 5151 (*d*) 5051

15. The value of $\binom{50}{1}^2+2\binom{50}{2}^2+3\binom{50}{3}^2+\ldots+50\binom{50}{50}^2$ is equal to (where $={}^nC_r$)

(*a*) $\binom{100}{50}$ (*b*) $50\left(\frac{100}{50}\right)$

(*c*) $100\binom{100}{50}$ (*d*) $\frac{2^{49}}{49!}(1.3.5\ldots99)$

16. We have $3m$ balls, out of which m are identical and the rest are all different. The number of ways of selecting m objects from the group is

(*a*) $\frac{{}^{2m}C_m}{2}$ (*b*) $\frac{1}{2}\,{}^{2m}C_{m-1}+2^{2m-1}$

(*c*) $\frac{1}{2}\,{}^{2m}C_m+2^{2m+1}$ (*d*) $\frac{1}{2}\,{}^{2m}C_m+2^{2m-1}$

17. Let $C_r's$ denote the combinational coefficient in the expansion of the binomial $(1+x)^{14}$, then the value of the sum $\Sigma_{r=1}^{15}r\cdot C_{r-1}$ is

(*a*) 2^{18} (*b*) 2^{17} (*c*) 2^{16} (*d*) 2^{15}

18. The value of $\Sigma_{r=1}^{15}r^2\cdot\left(\frac{{}^{26}C_r}{{}^{26}C_{r-1}}\right)$ is equal to

(*a*) 900 (*b*) 1100 (*c*) 1300 (*d*) 1800

19. Coefficient of α^t in the expansion of,
$(\alpha+p)^{(m-1)}+(\alpha+p)^{(m-2)}(\alpha+q)+(\alpha+p)^{(m-3)}(\alpha+q)^2+\ldots(\alpha+q)^{(m-1)}$ where $\alpha\ne-q$ and $p\ne q$ is

(*a*) $\frac{{}^mC_t(p^t-q^t)}{p-q}$ (*b*) $\frac{{}^mC_1(p^{m-t}-q^{m-t})}{p-q}$

(*c*) $\frac{{}^mC_t(p^t+q^t)}{p-q}$ (*d*) $\frac{{}^mC_t(p^{m-t}+q^{m-t})}{p-q}$

20. Sum of the series $\frac{1\cdot{}^{10}C_1}{2^9}-\frac{2\cdot{}^{10}C_2\cdot3^1}{2^8}+\frac{3\cdot{}^{10}C_3 3^2}{2^7}-\ldots-10\,{}^{10}C_{10}3^9$ is equal to $\frac{k}{2^8}$, then k is equal to

(*a*) 5^8 (*b*) -5^9 (*c*) -5^{10} (*d*) 5^{11}

21. The value of expression ${}^{(n+1)}C_2+3({}^2C_2+{}^3C_2+{}^4C_2+\ldots+{}^nC_2)$ is

(*a*) Σn^2 (*b*) Σn^2

(*c*) Σn (*d*) $\frac{n^2(n+1)}{2}$

22. Find the sum of following series

(*a*) $\sum_{r=0}^{n}{}^{2n}C_{2r}$ (*b*) $\sum_{r=0}^{25}{}^{100}C_{4r}$

(*c*) $\sum_{r=0}^{50}{}^{100}C_r$ (*d*) $\sum_{r=0}^{24}{}^{100}C_{4r+3}$

Important Results

If $(\sqrt{P}+Q)^n=I+f$ where I and n are positive integers, n being odd, and $0\le f<1$, then show that $(I+f)f=k^n$ where $P-Q^2=k>0$ and $\sqrt{P}-Q<1$.

Proof. Given $\sqrt{P}-Q<1$

$\therefore\ 0<(\sqrt{P}-Q)^n<1$

Now let $\left(\sqrt{P}-Q\right)^n=f'$ where $0<f'<1$

$\therefore\ I+f-f'=(\sqrt{P}+Q)^n-\left(\sqrt{P}-Q\right)^n$

$\because$ RHS contains even powers of $\sqrt{P}$ ($\because$ n is odd)

$\Rightarrow$ RHS is an integer

Since RHS and I are integers,

$\therefore$ $f-f'$ is also integer.

$\Rightarrow$ $f-f'=0$

$\because$ $-1<f-f'<1$

or $f=f'$

$\therefore$ $(I+f)f=(I+f)f'=(\sqrt{P}+Q)^n(\sqrt{P}-Q)^n$

$=(P-Q^2)^n=k^n$

If $(\sqrt{P}+Q)^n = I + f$ where I and n are positive integers, n being even, and $0 \le f < 1$, then show that $(I+f)(1-f) = k^n$ where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.

Proof: If n is an even integer then

$(\sqrt{P}+Q)^n + (\sqrt{P}-Q)^n = I + f + f'$

Hence, LHS and I are integers.

$\therefore$ $f + f'$ is also integer $\Rightarrow f + f' = 1$

$\because$ $0 < f + f' < 2$

$\therefore$ $f' = (1 - f)$

Hence, $(I+f)(1-f) = (I+f)f' = (\sqrt{P}+Q)^n(\sqrt{P}-Q)^n$

$= (P - Q^2)^n = k^n$.

Train Your Brain

Example 22: If $(2+\sqrt{3})^n = I + f$ where I and n are positive integers and $0 < f < 1$, show that

(*i*) I is an odd integer and

(*ii*) $(I+f)(1-f) = 1$.

Sol. (*i*) Now $0 < 2-\sqrt{3} < 1$, since $2-\sqrt{3} = 0.268$ (approx.)

$\therefore 0 < (2-\sqrt{3})^n < 1;$

we can take $(2-\sqrt{3})^n$ as f',

Now $(2+\sqrt{3})^n + (2-\sqrt{3})^n = I + f + f'$

But

L.H.S. $= 2(2^n + {}^nC_2 2^{n-2}(\sqrt{3})^2 + {}^nC_4 2^{n-4}(\sqrt{3})^4 + \dots)$

= an integer (in fact an even integer)

$\therefore$ RHS. $= I + f + f'$ = an even integer

Also $f + f' = 1$, since f and f' are both positive proper fractions.

$\therefore$ I = an even integer -1 = an odd integer.

(*ii*) $(I+f)(1-f) = (I+f)(f') = (2+\sqrt{3})^n.(2-\sqrt{3})^n$

$= (4-3)^n = 1^n = 1$

Concept Application

23. If $(9+\sqrt{80})^n = I + f$ where I, n are integers and $0 < f < 1$, then

(*a*) I is an odd integer

(*b*) I is an even integer

(*c*) $(I+f)(1-f) = 1$

(*d*) $1 - f = (9-\sqrt{80})^n$

DIVISIBILITY PROBLEMS

Working Rule

(*i*) First of all write down the given expression in such a way that there is a term containing n^{th} power of an integer a.

(*ii*) If not occurs, then go on subtracting 1, 2, 3, ... from a and find integer r such that some power of $(a-r)$ is divisible by the number k from which the given expression is to be shown to be divisible.

(*iii*) Now write $a^n = [r + (a-r)]^n$ and expand and then collect the terms containing $(a-r)^m$ and higher powers of $(a-r)$ in one bracket of $(a-r)^m$.

Train Your Brain

Example 23: Show that $3^{2n+2} - 8n - 9$ is divisible by 64 if $n \in N$.

Sol. $3^{2n+2} - 8n - 9 = (1+8)^{n+1} - 8n - 9$

$= [1 + (n+1)8 + {}^{n+1}C_2 8^2 + \dots] - 8n - 9$

$= {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + {}^{n+1}C_4 8^4 + \dots$

$= 64[{}^{n+1}C_2 + {}^{n+1}C_3 8 + {}^{n+1}C_4 8^2 + \dots]$

which is clearly divisible by 64.

Concept Application

24. The remainder, when $(15^{23} + 23^{23})$ is divided by 19, is

(*a*) 4 (*b*) 15 (*c*) 0 (*d*) 18

25. The remainder, if $1 + 2 + 2^2 + 2^3 + \dots + 2^{1999}$ is divided by 5 is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

26. Last two digits of 21 to the $(100)^{th}$ power, is

(*a*) 000 (*b*) 01 (*c*) 11 (*d*) 21

27. Last three digits of the number $N = 7^{100} - 3^{100}$ are

(*a*) 100 (*b*) 300 (*c*) 500 (*d*) 000

28. If $6^{83} + 8^{83}$ is divided by 49, then the remainder is

(*a*) 35 (*b*) 5 (*c*) 1 (*d*) 0

MULTINOMIAL EXPANSION/TRINOMIAL

If $n \in N$, then the general term of the multinomial expansion $(x_1 + x_2 + x_3 + \dots + x_k)^n$ is

$$\frac{n!}{a_1!a_2!a_3!\dots a_k!} x_1^{a_1}.x_2^{a_2}.x_3^{a_3}\dots x_k^{a_k}$$

where $a_1 + a_2 + a_3 + \dots + a_k = n$

and $0 \le a_i \le n$, $i = 1, 2, 3, \dots k$.

and the number of terms in the expansion are ${}^{n+k-1}C_{k-1}$.

The greatest coefficient in the expansion of $(a_1 + a_2 + a_3 + \dots + a_m)^n$

is $\dfrac{n!}{(q!)^{m-r}((q+1)!)^r}$

where q is the quotient and r is the remainder when n is divided by m.

Credit:--- Divansh.....

Train Your Brain

Example 24: Find the co-efficient of $a^2b^3c^5$ in the expansion of $(a + b - c)^{10}$.

Sol. General term is $\frac{10!}{\alpha_1!\alpha_2!\alpha_3!}(a)^{\alpha_1}(b)^{\alpha_2}(c)^{\alpha_3}$

$\therefore \quad \alpha_1 = 2, \alpha_2 = 3, \alpha_3 = 5$

$\therefore$ Co-efficient of $a^2b^3c^5 = \frac{10!}{2!3!5!}$.

Concept Application

29. The number of terms with rational coefficients in the expansion of $(\sqrt[4]{5x} + \sqrt[6]{3y} + z)^8$ is

(a) 7 (b) 6 (c) 8 (d) 9

30. The coefficient of term that contains $x^{10}y^3z^2$ expansion of the binomial $(x^2 - 2y + z)^{10}$ is

(a) $-\frac{8\cdot(10)!}{2!\cdot3!\cdot5!}$ (b) $\frac{8\cdot(10)!}{2!\cdot3!\cdot5!}$

(c) $\frac{(10)!}{2!\cdot3!\cdot5!}$ (d) $-\frac{(10)!}{2!\cdot3!\cdot5!}$

BINOMIAL THEOREM FOR ANY INDEX

If $n \in R$, $-1 < x < 1$, then

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots \infty$$

General term $t_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$

Note: (i) In the expansion if $n \in R$ and $n > 0$ then $-1 < x < 1$.

(ii) ${}^nC_r$ can not be used because it is defined only for natural number.

(iii) If x be so small then its square and higher powers may be neglected, then approximate value of $(1 + x)^n$ is $1 + nx$.

Important Results

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots + (-1)^r x^r + \dots$
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots + x^r + \dots$
- $(1 + x)^{-2} = 1 - {}^2C_1x + {}^3C_2x^2 - {}^4C_3x^3 + \dots + (-1)^r\,{}^{r+1}C_rx^r + \dots$
- $(1 - x)^{-2} = 1 + {}^2C_1x + {}^3C_2x^2 + {}^4C_3x^3 + \dots + {}^{r+1}C_rx^r + \dots$
- $(1 + x)^{-3} = 1 - {}^3C_1x + {}^4C_2x^2 - {}^5C_3x^3 + \dots + (-1)^r\,{}^{r+2}C_rx^r + \dots$
- $(1 - x)^{-3} = 1 + {}^3C_1x + {}^4C_2x^2 + {}^5C_3x^3 + \dots + {}^{r+2}C_rx^r + \dots$

In general, the coefficient of x^r in $(1 - x)^{-k}$ is ${}^{r+k-1}C_{k-1}$

APPROXIMATIONS

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \dots$$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its squares and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately.

This is an approximate value of $(1 + x)^n$

Train Your Brain

Example 25: If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}}$

Sol. $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1 - \frac{3}{2}x + 1 - \frac{5x}{3}}{2\left(1 + \frac{x}{4}\right)^{1/2}}$

$$= \frac{1}{2}\left(2 - \frac{19}{6}x\right)\left(1 + \frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2 - \frac{19}{6}x\right)\left(1 - \frac{x}{8}\right)$$

$$= \frac{1}{2}\left(2 - \frac{x}{4} - \frac{19}{6}x\right) = 1 - \frac{x}{8} - \frac{19}{12}x = 1 - \frac{41}{24}x$$

Example 26: The value of cube root of 1001 upto five decimal places is

Sol. $(1001)^{1/3} = (1000 + 1)^{\frac{1}{3}}$

$$= 10\left(1 + \frac{1}{1000}\right)^{\frac{1}{3}}$$

$$= 10\left\{1 + \frac{1}{3}\cdot\frac{1}{1000} + \frac{1/3(1/3-1)}{2!}\cdot\frac{1}{1000^2} + \dots\right\}$$

$$= 10\{1 + 0.0003333 - 0.0000001 + \dots\} = 10.00333$$

Concept Application

31. If the value of x is so small that x^2 and higher powers of x can be neglected, then $\frac{\sqrt{1+x} + \sqrt[3]{(1-x)^2}}{(1+x) + \sqrt{1+x}}$ is equal to:

(a) $1 + \frac{5}{6}x$ (b) $1 - \frac{5}{6}x$ (c) $1 + \frac{2}{3}x$ (d) $1 - \frac{2}{3}x$

32. If x is positive, the first negative term in the expansion of $(1 + x)^{27/5}$ is:

(a) 7th term (b) 5th term

(c) 8th term (d) 6th term

Credit:--- Divansh.....

Aarambh (Solved Examples)

1. If $(5+2\sqrt{6})^n = m + f$, where n and m are positive integers and $0 \le f < 1$, then $\frac{1}{1-f} - f$ is equal to

(a) $\frac{1}{m}$ (b) m (c) $m+\frac{1}{m}$ (d) $m-\frac{1}{m}$

Sol. We have $5-2\sqrt{6} = 1/\left(5+2\sqrt{6}\right)$.

Therefore, $0 < 5-2\sqrt{6} < 1$. Let $F = (5-2\sqrt{6})^n$. Then $0 < F < 1$. Also,

$m + f + F = (5+2\sqrt{6})^n + (5-2\sqrt{6})^n$

$= 2\left[C_0 5^n + C_2 5^{n-2}(2\sqrt{6})^2 + C_4 5^{n-4}(2\sqrt{6})^4 + \cdots\right] = 2k$

where k is some positive integer.

Hence, $f + F = 2k - m$ is a positive integer. Also, $0 < f + F < 2$. Therefore, $f + F = 1$. Now,

$\frac{1}{1-f} - f = \frac{1}{F} - (1-F)$

$= (5+2\sqrt{6})^n - \left\{1-(5-2\sqrt{6})^n\right\}$

$= (5+2\sqrt{6})^n + (5-2\sqrt{6})^n - 1$

$= m + f + F - 1 = m$

Therefore, option (*b*) is the correct answer.

2. If $S_1 = \sum_{k=0}^{n} {}^nC_k \cos(kx)\cos(n-k)x$ and

$S_2 = \sum_{k=1}^{n-1} {}^nC_k \sin(kx)\sin(n-k)x$, then $S_1 - S_2$ equals

(a) $2^{n-1}\cos(nx)$ (b) $2^n\cos(nx)$

(c) 0 (d) $2+2^n\cos nx$

Sol. $S_1 - S_2 = \sum_{k=0}^{n} {}^nC_k[\cos(kx)\cos(n-k)x - \sin(kx)\sin(n-k)x]$

$= \sum_{k=0}^{n} {}^nC_k \cos(k+n-k)x$

$= 2^n\cos(nx)$

Therefore, option (*b*) is the correct answer.

3. If $\left(1+x+x^2\right)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, then value of $N = a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + \cdots$ is equal to

(a) 0 (b) n (c) $-n$ (d) 1

Sol. N = Coefficient of $\frac{1}{x}$ in

$\left(a_0 - a_1x + a_2x^2 \cdots + a_{2n}x^{2n}\right)$

$\times\left(a_0 + a_1\left(\frac{1}{x}\right) + a_2\left(\frac{1}{x}\right)^2 + \cdots a_{2n}\left(\frac{1}{x}\right)^{2n}\right.$

$=$ coefficient of $\frac{1}{x}$ in $\left(1-x+x^2\right)^n$

$\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n$ or in $\frac{1}{x^{2n}}\left[\left(1+x^2\right)^2 - x^2\right] = 0$

Therefore, option (*a*) is the correct answer.

4. Value of $S = \sum_{k=1}^{\infty}\sum_{r=0}^{k}\frac{1}{3^k}\left({}^kC_r\right)$ is

(a) 2 (b) $\frac{2}{3}$

(c) $\frac{1}{3}$ (d) None of these

Sol. $\sum_{r=0}^{k}\frac{1}{3^k}\left({}^kC_r\right) = \frac{1}{3^k}\sum_{r=0}^{k}{}^kC_r = \frac{2^k}{3^k}$

Thus, $S = \frac{2/3}{1-2/3} = 2$

Therefore, option (*a*) is the correct answer.

5. The coefficient of term that contains $x^{10}y^3z^2$ in the expansion of the binomial $(x^2-2y+z)^{10}$ is

(a) $-\frac{8\cdot(10)!}{2!\cdot 3!5!}$ (b) $\frac{8\cdot(10)!}{2!\cdot 3!\cdot 5!}$

(c) $\frac{(10)!}{2!\cdot 3!\cdot 5!}$ (d) $-\frac{(10)!}{2!\cdot 3!\cdot 5!}$

Sol. Consider $(x^2+(z-2y))^{10}$ and using multinomial coefficent

$= -\frac{8\cdot(10)!}{2!\cdot 3!5!}$.

Therefore, option (*a*) is the correct answer.

6. If $2a+2b+3c = \frac{1}{5}$ and $a, b, c > 0$ then maximum value of term independent of x in the expansion of $\left(abx^{\frac{1}{2}} + cx^{\frac{-1}{3}}\right)^{25}$ is:

(a) ${}^{25}C_{10}$ (b) ${}^{25}C_{15}(25)^{25}$

(c) ${}^{25}C_{15}\left(\frac{1}{35}\right)^{35}$ (d) ${}^{25}C_{10}\left(\frac{1}{25}\right)^{35}$

Sol. $T_{r+1} = {}^{25}C_r(ab)^{(k-r)} \cdot x^{\left(\frac{k-r}{2}\right)} \cdot c^r \cdot x^{\left(\frac{-r}{3}\right)}$

$k = 25$

$\Rightarrow r = 15 = {}^{25}C_{15}\, a^{10}\cdot b^{10}\cdot c^{15}$

$10a + 10b + 15c = 1$

$\frac{(a+a+\ldots+a)+(b+b+\ldots+b)+(c+c+\ldots+c)}{35}$

$\ge \left(a^{10}b^{10}c^{15}\right)^{\frac{1}{35}} \Rightarrow \left(\frac{1}{35}\right)^{35} \ge a^{10}b^{10}c^{15}$

Therefore, option (*c*) is the correct answer.

7. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2 + (1+x)^3 + \ldots\ldots + (1+x)^{49} + (1+mx)^{50}$ is $(3n+1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is:

(a) 1 (b) 2 (c) 4 (d) 5

Sol. ${}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots. + {}^{49}C_2 + {}^{50}C_2\, m^2 = (3n+1)\,{}^{51}C_3$

${}^3C_3 + {}^3C_2 + {}^4C_2 + \ldots. + {}^{49}C_2 + {}^{50}C_2\, m^2 = (3n+1)\,{}^{51}C_3$

${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \Rightarrow {}^{50}C_3 + {}^{50}C_2 \cdot m^2 = (3n+1)\,{}^{51}C_3$

${}^{50}C_3 + {}^{50}C_2 + (m^2-1)\,{}^{50}C_2 = 3n.\frac{51}{3}.{}^{50}C_2 + {}^{51}C_3$

$\Rightarrow {}^{51}C_3 + (m^2-1)\,{}^{50}C_2 = 51n\cdot{}^{50}C_2 + {}^{51}C_3$

$m^2 - 1 = 51n \Rightarrow \quad m^2 = 51n + 1$

min value of m^2 for $51n+1$ is integer for $n = 5$

Therefore, option (*d*) is the correct answer.

8. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots. + C_nx_n$ then $\sum_{0\le i<j\le n}(C_i+C_j)^2 =$

(a) $(n-1)\,{}^{2n}C_n + 2^{2n}$ (b) $(n+2)\,{}^{2n}C_n + 3^{2n}$

(c) $(n+3)\,{}^{2n}C_n + 2^{2n}$ (d) $(n-4)\,{}^{2n}C_n + 5^{2n}$

Sol. $\sum\sum_{0\le i<j\le n}\left(C_i+C_j\right)^2$

$(C_0+C_1)^2 + (C_0+C_2)^2 + \ldots + (C_0+C_n)^2 + (C_1+C_3)^2 + \ldots + (C_1+C_n)^2 + (C_2+C_3)^2 + (C_2+C_4)^2 + \ldots + (C_2+C_n)^2 + \ldots + (C_{n-1}+C_n)^2$

$= n\left(C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2\right) + 2\sum\sum_{0\le i<j\le n} C_iC_j$

$= n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \frac{2n!}{2.n!n!}\right\}$

$n.{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n-1).{}^{2n}C_n + 2^{2n}$

Therefore, option (*a*) is the correct answer.

9. ${}^{25}C_{10} + {}^{24}C_{10} + \ldots\ldots\ldots + {}^{10}C_{10} =$

(a) ${}^{23}C_{11}$ (b) ${}^{26}C_{11}$

(c) ${}^{25}C_{11}$ (d) ${}^{22}C_{11}$

Credit:--- Divansh.....

Sol. ${}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \dots + {}^{25}C_{10}$

$\Rightarrow {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \dots + {}^{25}C_{10}$

$\Rightarrow {}^{12}C_{11} + {}^{12}C_{10} + \dots + {}^{25}C_{10}$

$\Rightarrow {}^{13}C_{11} + {}^{13}C_{10} + \dots + {}^{25}C_{10}$

and so on $\therefore\ {}^{26}C_{11}$

Alternate: Coefficient of x^{10} in

$\{(1+x)^{10} + (1+x)^{11} + \dots (1+x)^{25}\}$

$\Rightarrow$coefficient of x^{10} in $\left[(1+x)^{10}\dfrac{(1+x)^{16}-1}{1+x-1}\right]$

$\Rightarrow$coefficient of x^{10} in $\dfrac{\left[(1+x)^{26}-(1+x)^{10}\right]}{x}$

$\Rightarrow$coefficient of x^{11} in $[(1+x)^{26}-(1+x)^{10}]$.

$= {}^{26}C_{11} - 0 = {}^{26}C_{11}$

Therefore, option (*b*) is the correct answer.

10. Find coefficient of $x^5y^4z^3$ in $(x+y+z)^{12}$.

(*a*) $\dfrac{12!}{5!\,4!\,3!}$ (*b*) $\dfrac{11!}{3!4!5!}$

(*c*) $\dfrac{9!}{4!3!5!}$ (*d*) $\dfrac{10!}{5!4!3!}$

Sol. $[x+y+z]^{12} = [x+(y+z)]^{12}$

$T_{r+1} = {}^{12}C_r \cdot x^{12-r} \cdot (y+z)^r$

put $r = 7$

$T_8 = {}^{12}C_7 \cdot x^5 \cdot (y+z)^7 = {}^{12}C_7 \cdot x^5 \cdot {}^7C_p \cdot y^{7-p} \cdot z^p$

put $p = 3 = {}^{12}C_7 \cdot {}^7C_3 \cdot x^5 \cdot y^4 \cdot z^3$

Hence coefficient $x^5y^4z^3$ is ${}^{12}C_7 \cdot {}^7C_3 = \dfrac{(12)!\ \ 7!}{7!\cdot 5!\,3!\cdot 4!} = \dfrac{(12)!}{3!\cdot 4!\cdot 5!}$

Therefore, option (*a*) is the correct answer.

11. Suppose $\alpha, \beta > 0$ and $x_n = {}^nC_0\alpha^n + {}^nC_2\alpha^{n-2}\beta^2 + \cdots$

$y_n = {}^nC_1\alpha^{n-1}\beta + {}^nC_3\alpha^{n-3}\beta^3 + \cdots$ then

(*a*) $2x_n = (\alpha+\beta)^n + (\alpha-\beta)^n$

(*b*) $2y_n = (\alpha+\beta)^n - (\alpha-\beta)^n$

(*c*) $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n} = 1$

(*d*) $\lim\limits_{n\to\infty} e^{x_n/y_n} = e$

Sol. $x_n + y_n = (\alpha+\beta)^n$

$x_n - y_n = (\alpha-\beta)^n \Rightarrow 2x_n = (\alpha+\beta)^n + (\alpha-\beta)^n$

and $2y_n = (\alpha+\beta)^n - (\alpha-\beta)^n$

Credit:--- Divansh.....

$$\Rightarrow \frac{x_n}{y_n}=\frac{(\alpha+\beta)^n+(\alpha-\beta)^n}{(\alpha+\beta)^n-(\alpha-\beta)^n}=\frac{1+\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^n}{1-\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^n}$$

As $-(\alpha+\beta)<\alpha-\beta<\alpha+\beta$

$$\Rightarrow -1<\frac{\alpha-\beta}{\alpha+\beta}<1 \Rightarrow \lim_{n\to\infty}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^n=0$$

$$\therefore \lim_{n\to\infty}\frac{x_n}{y_n}=1\Rightarrow \lim_{n\to\infty}e^{x_n/y_n}=e$$

Therefore, option (*a, b, c, d*) is the correct answer.

12. If $(1+2x+3x^2)^{10}=a_0+a_1x+a_2x^2+\cdots+a_{20}x^{20}$, then

(*a*) $a_1=20$ (*b*) $a_2=1210$

(*c*) $a_4=6085$ (*d*) $a_{20}=2^2\times 3^9\times 7$

Sol. $(1+2x+3x^2)^{10}$

$= 1 + 10(2x + 3x^2) + {}^{10}C_2(2x + 3x^2)^2 + \ldots + {}^{10}C_{10}(2x+3x^2)^{10}$

$\Rightarrow a_1 = 20,\ a_2 = 30 + (45)(4) = 210$

$a_4 = {}^{10}C_2(3^2) + {}^{10}C_3\,({}^3C_1)\,(2^2)\,(3) + {}^{10}C_4(2^4) = 8085$

$a_{20} = 3^{10}$

Therefore, option (*a*) is the correct answer.

13. Let $f(n)=\Sigma_{k=1}^{n}k^2C_k^2$ then $\dfrac{f(5)}{875}$ is

Sol. $k(C_k)=k\dfrac{n!}{k!(n-k)!}=n\left({}^{n-1}C_{k-1}\right)$

Thus, $\Sigma_{k=1}^{n}k^2C_k^2=n^2\,\Sigma_{k=1}^{n}\left({}^{n-1}C_{k-1}\right)^2$

$$=n^2\sum_{k=0}^{n-1}\left({}^{n-1}C_k\right)^2$$

But $\Sigma_{k-0}^{n}C_k^2={}^{2n}C_n$, therefore,

$$\Sigma_{k=1}^{n}k^2C_k^2=n^2\left({}^{2n-2}C_{n-1}\right)$$

$$\Rightarrow \frac{f(5)}{875}=\frac{5^2\left({}^8C_4\right)}{875}=2$$

Therefore, [2] is the correct answer.

14. For $n > 2$, let $C_r = {}^nC_r$, and $f(n) = 3.C_2 + 5.C_4 + 7.C_6 + \ldots$, then $2^{-5}f(6)$ equals

Sol. We have

$$C_0+C_2x^2+C_4x^4+\ldots=\frac{1}{2}\left[(1+x)^n+(1-x)^n\right]$$

$$\Rightarrow C_0x+C_2x^3+C_4x^5+\ldots=\frac{1}{2}\left[(1+x)^nx+(1-x)^nx\right]$$

Differentiating both the sides w.r.t. x, we get

$$C_0+3C_2x^2+5C_4x^4+\ldots$$

$$=\frac{1}{2}\left[(1+x)^n+n(1+x)^{n-1}x+(1-x)^n-nx(1-x)^{n-1}\right]$$

Putting $x = 1$, we get

$$f(n)=C_0+3C_2+5C_4+\ldots=2^{n-1}+n\left(2^{n-2}\right)$$

$$\therefore 2^{-5}f(6)=4$$

Therefore, [4] is the correct answer.

15. If $\left(x+\dfrac{1}{x}+1\right)^6=a_0+\left(a_1x+\dfrac{b_1}{x}\right)+\left(a_2x^2+\dfrac{b_2}{x^2}\right)+\ldots+\left(a_6x^6+\dfrac{b_6}{x^6}\right)$, then find the value of a_0.

Sol. $\left(x+\dfrac{1}{x}+1\right)^6=\Sigma_{r=1}^{6}\,{}^6C_r\left(x+\dfrac{1}{x}\right)^r$

$$\therefore a_0=1+{}^6C_2\cdot{}^2C_1+{}^6C_4\cdot{}^4C_2+{}^6C_6\cdot{}^6C_3$$

$= 1 + 30 + 90 + 20 = 141$

Therefore, [141] is the correct answer.

> "An optimist studies the toughest chapter thinking: 'One more question I'll ace on the exam!"

Credit:--- Divansh.....

School Level Problems

SINGLE CORRECT TYPE QUESTIONS

1. The total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification is

(*a*) 50 (*b*) 202
(*c*) 51 (*d*) None of these

2. The sum of the series $\sum_{r=0}^{10} {}^{20}C_r$ is

(*a*) $2^{19}+\frac{{}^{20}C_{10}}{2}$ (*b*) $\frac{{}^{20}C_{10}}{2}$

(*c*) $2^{19}-\frac{{}^{20}C_{10}}{2}$ (*d*) $2^{19}+{}^{20}C_{10}$

3. The coefficient of x^{20} in the expansion of $(1 + x)^{40}$ and $(1 + x)^{39}$ are in the ratio.

(*a*) 1 : 2 (*b*) 1 : 3
(*c*) 3 : 1 (*d*) 2 : 1

4. If x^m occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{2n}$, the coefficient of x^m is

(*a*) $\frac{(2n)!}{\left(\frac{2n-m}{3}\right)!\left(\frac{4n+m}{3}\right)!}$

(*b*) $\frac{(2n)!}{m!(2n-m)!}$

(*c*) $\frac{(2n)!3!3!}{(2n-m)!}$

(*d*) None of these

5. If A and B are coefficient of x^n in the expansions of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then $\frac{A}{B}$ equals

(*a*) 1 (*b*) 2

(*c*) $\frac{1}{2}$ (*d*) $\frac{1}{n}$

6. If the middle term of $\left(\frac{1}{x}+x\sin x\right)^{10}$ is equal to $7\frac{7}{8}$, then value of x is

(*a*) $2n\pi+\frac{\pi}{6}$ (*b*) $n\pi+\frac{\pi}{6}$

(*c*) $n\pi+(-1)^n\frac{\pi}{6}$ (*d*) $n\pi+(-1)^n\frac{\pi}{3}$

7. The value of ${}^{47}C_4+\sum_{r=1}^{5}{}^{52-r}C_3$ is equal to

(*a*) ${}^{47}C_6$ (*b*) ${}^{52}C_5$
(*c*) ${}^{52}C_4$ (*d*) None of these

8. Given the integers $r > 1, n > 2$, and coefficients of $(3r)^{th}$ and $(r + 2)^{nd}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal, then

(*a*) $n = 2r$ (*b*) $n = 3r$
(*c*) $n = 2r + 1$ (*d*) None of these

9. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then equals $\frac{(C_0+C_1)(C_1+C_2)\dots(C_{n-1}+C_n)}{C_1C_2\dots C_n}$.

(*a*) $\frac{(n+1)^n}{n!}$ (*b*) $\frac{n^n}{(n+1)!}$

(*c*) $\frac{n^n}{n!}$ (*d*) None of these

VERY SHORT ANSWER TYPE QUESTIONS

10. Find the coefficient of x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$.

11. If the coefficient of second, third and fourth terms in the expansion of $(1 + x)^{2n}$ are in A.P. Show that $2n^2 - 9n + 7 = 0$.

12. Find the middle term (terms) in the expansion of

(i) $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$

(ii) $\left(3x-\frac{x^3}{6}\right)^9$

13. Find the coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$.

14. Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$.

15. Find the sixth term of the expansion $\left(y^{\frac{1}{2}}+x^{\frac{1}{3}}\right)^n$, if the binomial coefficient of the third term from the end is 45.

16. If the term free from x in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is 405, find the value of k.

LONG ANSWER TYPE QUESTIONS

17. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, find p.

18. If x^p occurs in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$, prove that its coefficient is

$$\frac{2n!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$$

Credit:--- Divansh.....

19. Find n in the binomial $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$ if the ratio of 7th term from the beginning to the 7th term from the end is $\frac{1}{6}$.

20. Show that the middle term in the expansion of $\left(x-\frac{1}{x}\right)^{2n}$ is $\frac{1\times3\times5\times\ldots(2n-1)}{n!}\times(-2)^n$.

21. Find the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

CASE STUDY BASED QUESTIONS

Case Study-I

The binomial theorem is a fundamental concept in algebra, enabling the expansion of expressions in the form $(a+b)^n$. In this case study, students are tasked with determining a particular 'x' term within such an expansion.

22. In the expansion of $(2x-3y)^5$, what is the term that contains 'x'?

(a) The 3rd term (b) The 4th term

(c) The 5th term (d) The 6th term

23. If you were to expand $(2x-3y)^5$ completely, how many terms would there be in the expansion?

(a) 5 (b) 6

(c) 10 (d) 15

Case Study-II

In binomial theorem, determining the middle term of an expansion can be a valuable skill. Students are challenged to find the middle term in a complex expansion.

24. In the expansion of $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$, which term represents the middle term?

(a) The 3rd term

(b) The 4th terms

(c) The 5th term

(d) The 6th term

25. What is the coefficient of the middle term in the expansion $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$?

(a) 20 (b) –20 (c) 36 (d) –36

Prarambh (Topicwise)

GENERAL TERM

1. 6th term in expansion of $\left(2x^2-\frac{1}{3x^2}\right)^{10}$ is

(a) $\frac{4580}{17}$ (b) $-\frac{896}{27}$

(c) $\frac{5580}{17}$ (d) $\frac{-764}{25}$

2. If the coefficients of r^{th} term and $(r+4)^{th}$ term are equal in the expansion of $(1+x)^{20}$, then the value of r will be

(a) 7 (b) 8

(c) 9 (d) 10

3. 16th term in the expansion of $(\sqrt{x}-\sqrt{y})^{17}$ is

(a) $136xy^7$ (b) $136xy$

(c) $-136xy^{15/2}$ (d) $-136xy^2$

4. If x^4 occurs in the r^{th} term in the expansion of $\left(x^4+\frac{1}{x^3}\right)^{15}$, then $r=$

(a) 7 (b) 8 (c) 9 (d) 10

5. If the third term in the binomial expansion of $(1+x)^m$ is $-\frac{1}{8}x^2$, then the rational value of m is

(a) 2 (b) 1/2 (c) 3 (d) 4

6. Coefficient of x in the expansion of $\left(x^2+\frac{a}{x}\right)^5$ is

(a) $9a^2$ (b) $10a^3$ (c) $10a^2$ (d) $10a$

7. In the expansion of $\left(x-\frac{1}{x}\right)^6$, the constant term is

(a) –20 (b) 20 (c) 30 (d) –30

8. In the expansion of $(x^2-2x)^{10}$, the coefficient of x^{16} is

(a) –1680 (b) 1680 (c) 3360 (d) 6720

9. The coefficient of x^{32} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ is

(a) ${}^{15}C_5$ (b) ${}^{15}C_6$ (c) ${}^{15}C_4$ (d) ${}^{15}C_7$

10. If the coefficients of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55

(c) 45 (d) 15

11. In the expansion of $(1+3x+2x^2)^6$ the coefficient of x^{11} is

(a) 144 (b) 288

(c) 216 (d) 576

Credit:--- Divansh.....

12. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$, which of the following statements is false.

(a) The number of irrational terms is 19.

(b) Middle term is rational.

(c) The number of rational terms is 2.

(d) 9th term is rational.

13. In the expansion of $\left(x^{2/3}-\frac{1}{\sqrt{x}}\right)^{30}$, a term containing the power x^{14}

(a) Does not exist

(b) Exists & the co-efficient is divisible by 29

(c) Exists & the co-efficient is divisible by 63

(d) Exists & the co-efficient is divisible by 65

14. In the expansion of $\left(x^3+3.2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$, which of the following statements is false.

(a) There appears a term with the power x^2

(b) There does not appear a term with the power x^2

(c) There appears a term with the power x^{-3}

(d) The ratio of the co-efficient of x^3 to that of x^{-3} is 1/3

15. If the 6th term in the expansion of $\left(\frac{3}{2}+\frac{x}{3}\right)^n$ when $x = 3$ is numerically greatest then the possible integral values of n can not be

(a) 11 (b) 12 (c) 13 (d) 14

TERM INDEPENDENT OF 'X'

16. In the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$, the term independent of x is

(a) ${}^9C_3\cdot\frac{1}{6^3}$ (b) ${}^9C_3\left(\frac{3}{2}\right)^3$

(c) ${}^9C_3$ (d) ${}^9P_2\left(\frac{3}{2}\right)^3$

17. The term independent of x in the expansion of $\left(x^2-\frac{3\sqrt{3}}{x^3}\right)^{10}$ is

(a) 153090 (b) 150000

(c) 150090 (d) 153180

18. The term independent of x in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$ is

(a) $C_0^2+2C_1^2+...+(n+1)C_n^2$

(b) $(C_0+C_1+...+C_n)^2$

(c) $C_0^2+C_1^2+...+C_n^2$

(d) $C_0^2+2C_1^2+...+nC_n^2$

MIDDLE TERM/NUMERICALLY GREATEST TERM AND ALGEBRICALLY GREATEST & LEAST TERM

19. If the middle term in the expansion of $\left(x^2+\frac{1}{x}\right)^n$ is $924x^6$, then $n =$

(a) 10 (b) 12 (c) 14 (d) 16

20. The middle term in the expansion of $(1+x)^{2n}$ is

(a) $\frac{(2n)!}{n!}x^{2n}$ (b) $\frac{(2n)!}{n!(n-1)!}x^{n+1}$

(c) $\frac{(2n)!}{(n!)^2}x^n$ (d) $\frac{(2n)!}{(n+1)!(n-1)!}x^n$

21. The middle term in the expansion of $\left(x+\frac{1}{2x}\right)^{2n}$ is

(a) $\frac{1.3.5....(2n-3)}{n!}$ (b) $\frac{1.3.5....(2n-1)}{n!}$

(c) $\frac{1.3.5....(2n+1)}{n!}$ (d) $\frac{1.3.5....(2n-3)}{n!}$

PROPERTIES OF BINOMIAL COEFFICIENT

22. $C_0C_r+C_1C_{r+1}+C_2C_{r+2}+...+C_{n-r}C_n =$

(a) $\frac{(2n)!}{(n-r)!(n+r)!}$ (b) $\frac{n!}{(2n-r)!(n-r)!}$

(c) $\frac{n!}{(n-r)!}$ (d) $\frac{2n!}{(n-r)!}$

23. $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+...+\frac{C_n}{n+1}=$

(a) $\frac{2^n}{n+1}$ (b) $\frac{2^n-1}{n+1}$

(c) $\frac{2^{n+1}-1}{n+1}$ (d) $\frac{2^{n-1}-1}{n-1}$

24. The sum of all the coefficients in the binomial expansion of $(x^2+x-3)^{319}$ is

(a) 1 (b) 2

(c) −1 (d) 0

25. The sum of the coefficients of even power of x in the expansion of $(1+x+x^2+x^3)^5$ is

(a) 256 (b) 128

(c) 512 (d) 64

26. The value of ${}^{15}C_1^2-{}^{15}C_2^2+{}^{15}C_3^2-...-{}^{15}C_{15}^2$ is

(a) 15 (b) −15 (c) 0 (d) 51

APPLICATION OF BINOMIAL THEOREM

27. The greatest integer which divides the number $101^{100}-1$, is

(a) 100 (b) 1000

(c) 10000 (d) 100000

28. The last digit in 7^{300} is

(a) 7 (b) 9 (c) 1 (d) 3

29. The number 111 ... 1 (91 times) is

(a) Not a prime (b) An even number

(c) Not an odd number (d) Prime Number

30. The number $101^{100} - 1$ is not divisible by

(a) 50 (b) 500

(c) 5000 (d) 100000

31. $7^9 + 9^7$ is divisible by:

(a) 16 (b) 24

(c) 65 (d) 72

MULTINOMIAL THEOREM AND OTHER INDEX

32. In the expansion of $\left(\frac{1+x}{1-x}\right)^2$, the coefficient of x^n will be [where $|x| < 1$]

(a) $4n$ (b) $4n-3$

(c) $4n+1$ (d) $4n-1$

33. The coefficient of two consecutive terms in the expansion of $(1+x)^n$ will be equal, if

(a) n is any integer (b) n is an odd integer

(c) n is an even integer (d) n is prime number

NUMERICAL FRACTION

34. Let $x = (5\sqrt{2}+7)^{19}$, then $x.\{x\}$ ($\{.\}$ denotes the fractional part of x) is equal to

(a) 2^{19} (b) 3^{19} (c) 0 (d) 1

35. If $(4+\sqrt{15})^n = I + f$, where n is an odd natural number, I is an integer and $0 < f < 1$, then which of the following statements is false.

(a) I is natural number (b) I is an even integer

(c) $(I+f)(1-f) = 1$ (d) $1-f = (4-\sqrt{15})^n$

36. The total number of terms in the expansion of, $(x+4)^{100} + (x-4)^{100}$ after simplification is:

(a) 50 (b) 202 (c) 51 (d) 49

37. If the second term of the expansion $\left[a^{1/13} + \frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\frac{{}^nC_3}{{}^nC_2}$ is:

(a) 4 (b) 3 (c) 12 (d) 6

38. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$; the number of terms free from radicals is:

(a) 730 (b) 729 (c) 725 (d) 750

39. The value of m, for which the coefficients of the $(2m+1)^{\text{th}}$ and $(4m+5)^{\text{th}}$ terms in the expansion of $(1+x)^{10}$ are equal, is

(a) 3 (b) 1 (c) 5 (d) 8

40. The greatest terms of the expansion $(2x+5y)^{13}$ when $x = 10, y = 2$ is

(a) ${}^{13}C_5 . 20^8 . 10^5$ (b) ${}^{13}C_6 . 20^7 . 10^4$

(c) ${}^{13}C_4 . 20^9 . 10^4$ (d) ${}^{13}C_2 . 20^6 . 10^5$

41. Find numerically greatest term in the expansion of $(2+3x)^9$, when $x = 3/2$.

(a) ${}^9C_6 . 2^9 . (3/2)^{12}$ (b) ${}^9C_3 . 2^9 . (3/2)^6$

(c) ${}^9C_5 . 2^9 . (3/2)^{10}$ (d) ${}^9C_4 . 2^9 . (3/2)^8$

42. The remainder when 2^{2003} is divided by 17 is:

(a) 1 (b) 2 (c) 8 (d) 11

43. The greatest integer less than or equal to $(\sqrt{2}+1)^6$ is

(a) 196 (b) 197 (c) 198 (d) 199

Prabal (JEE Main Level)

SINGLE CORRECT TYPE QUESTIONS

1. The $(m+1)^{\text{th}}$ term of $\left(\frac{x}{y}+\frac{y}{x}\right)^{2m+1}$ is:

(a) Independent of x

(b) A constant

(c) Depends on the ratio x/y and m

(d) Independent of y

2. If the 6^{th} term in the expansion of the binomial $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ is 5600, then $x =$

(a) 10 (b) 8

(c) 11 (d) 9

3. The co-efficient of x in the expansion of $(1-2x^3+3x^5)\left(1+\frac{1}{x}\right)^8$ is:

(a) 56 (b) 65 (c) 154 (d) 62

4. The term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^3$ is:

(a) −3 (b) 0 (c) 1 (d) 3

5. If the coefficients of x^7 & x^8 in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is

(a) 15 (b) 45 (c) 55 (d) 56

Credit:--- Divansh.....

6. Number of rational terms in the expansion of $(\sqrt{2}+\sqrt[4]{3})^{100}$ is

(*a*) 25 (*b*) 26

(*c*) 27 (*d*) 28

7. If $k \in R$ and the middle term of $\left(\frac{k}{2}+2\right)^8$ is 1120, then value of k is:

(*a*) 3 (*b*) 2 (*c*) -3 (*d*) -4

8. The sum of the binomial coefficients of $\left[2x+\frac{1}{x}\right]^n$ is equal to 256. The constant term in the expansion is

(*a*) 1120 (*b*) 2110 (*c*) 1210 (*d*) 1102

9. The sum of the co-efficients in the expansion of $(1-2x+5x^2)^n$ is '*a*' and the sum of the co-efficients in the expansion of $(1+x)^{2n}$ is b. Then

(*a*) $a=b$ (*b*) $a=b^2$ (*c*) $a^2=b$ (*d*) $ab=1$

10. The binomial expansion of $\left(x^k+\frac{1}{x^{2k}}\right)^{3n}$, $n \in N$ contains a term independent of x

(*a*) Only if k is an integer

(*b*) Only if k is a natural number

(*c*) Only if k is rational

(*d*) For any real k

11. If the expansion of $(3x+2)^{-1/2}$ is valid in ascending powers of x, then x lies in the interval.

(*a*) $(-2/3, 2/3)$ (*b*) $(-3/2, 3/2)$

(*c*) $(-1/3, 2/3)$ (*d*) $(-\infty, -3/2) \cup (3/2, \infty)$

12. Coefficient of α^t in the expansion of $(\alpha+p)^{m-1} + (\alpha+p)^{m-2}(\alpha+q) + (\alpha+p)^{m-3}(\alpha+q)^2 + \ldots (\alpha+q)^{m-1}$ where $\alpha \neq -q$ and $p \neq q$ is

(*a*) $\frac{{}^mC_t(p^t-q^t)}{p-q}$ (*b*) $\frac{{}^mC_t(p^{m-t}-q^{m-t})}{p-q}$

(*c*) $\frac{{}^mC_t(p^t+q^t)}{p-q}$ (*d*) $\frac{{}^mC_t(p^{m-t}+q^{m-t})}{p-q}$

13. The numerically greatest term in the expansion of $(2x+5y)^{34}$, when $x=3$ & $y=2$ is:

(*a*) T_{21} (*b*) T_{22} (*c*) T_{23} (*d*) T_{24}

14. $\sum_{r=0}^{n-1} \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} =$

(*a*) $\frac{n}{2}$ (*b*) $\frac{n+1}{2}$

(*c*) $(n+1)\frac{n}{2}$ (*d*) $\frac{n(n-1)}{2(n+1)}$

15. Set of values of r for which, ${}^{18}C_{r-2}+2.{}^{18}C_{r-1}+{}^{18}C_r \geq {}^{20}C_{13}$ contains

(*a*) 4 elements (*b*) 5 elements

(*c*) 7 elements (*d*) 10 elements

16. $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_{10}}{11}=$

(*a*) $\frac{2^{11}}{11}$ (*b*) $\frac{2^{11}-1}{11}$

(*c*) $\frac{3^{11}}{11}$ (*d*) $\frac{3^{11}-1}{11}$

17. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to:

(*a*) ${}^{47}C_5$ (*b*) ${}^{52}C_5$ (*c*) ${}^{52}C_4$ (*d*) ${}^{49}C_4$

18. The value of $\binom{50}{0}\binom{50}{1}+\binom{50}{1}\binom{50}{2}+\ldots+\binom{50}{49}\binom{50}{50}$ is,

where ${}^nC_r = \binom{n}{r}$

(*a*) $\binom{100}{50}$ (*b*) $\binom{100}{51}$ (*c*) $\binom{50}{25}$ (*d*) $\binom{50}{25}^2$

19. The last two digits of the number 3^{400} are:

(*a*) 81 (*b*) 43 (*c*) 29 (*d*) 01

20. The co-efficient of x^4 in the expansion of $(1-x+2x^2)^{12}$ is

(*a*) ${}^{12}C_3$ (*b*) ${}^{13}C_3$

(*c*) ${}^{14}C_4$ (*d*) ${}^{12}C_3+3\,{}^{13}C_3+{}^{14}C_4$

21. The value of ${}^nC_0 \,.\, {}^nC_n + {}^nC_1 \,.\, {}^nC_{n-1} + \ldots + {}^nC_n \,.\, {}^nC_0$ is

(*a*) ${}^{2n}C_{n-1}$ (*b*) ${}^{2n}C_n$ (*c*) ${}^{2n}C_{n+1}$ (*d*) ${}^{2n}C_2$

22. The value of r for which ${}^{30}C_r \,.\, {}^{20}C_0 + {}^{30}C_{r-1} \,.\, {}^{20}C_1 + \ldots + {}^{30}C_0 \cdot {}^{20}C_r$ is maximum, is

(*a*) 10 (*b*) 15 (*c*) 20 (*d*) 25

23. The value of p, for which coefficient of x^{50} in the expression $(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+\ldots+1001\,x^{1000}$ is equal to ${}^{1002}C_p$, is:

(*a*) 10 (*b*) 50

(*c*) 20 (*d*) 25

24. The number of values of 'r' satisfying the equation, ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$ is

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4

25. 2^{60} when divided by 7 leaves the remainder equal to

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

26. Let $S_n = 1+q+q^2+\ldots+q^n$ and $T_n = 1+\left(\frac{q+1}{2}\right)+\left(\frac{q+1}{2}\right)^2+\ldots+\left(\frac{q+1}{2}\right)^n$ where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2 . S_1 + \ldots + {}^{101}C_{101} . S_{100} = \alpha T_{100}$ then α is equal to:

(*a*) 2^{99} (*b*) 202

(*c*) 200 (*d*) 2^{100}

Credit:--- Divansh.....

27. The sum $\sum_{l=0}^{m}\binom{10}{l}\binom{20}{m-l}$, (where $\binom{p}{q}=0$ if $p<q$) is maximum when m is

(a) 10 (b) 15 (c) 20 (d) 25

28. How many terms in the expansion of $\left(y^{\frac{1}{3}}+x^{\frac{1}{10}}\right)^{55}$ are free from radical sign?

(a) 1 (b) 2 (c) 3 (d) 6

29. Let k be the coefficient of x^{18} in the polynomial $f(x) = (1+x)^{20}+x(1+x)^{19}+x^2(1+x)^{18}+.+x^{18}(1+x)^2$ then the value of k is equal to

(a) 1300 (b) 1310

(c) 1330 (d) 1340

30. If $\sum_{r-1}^{10} r(r-1)^{10}C_r = k\cdot 2^9$, then k is equal to

(a) 10 (b) 45 (c) 90 (d) 100

31. The sum of the coefficients of even power of x in the expansion of $(1+x+x^2+x^3)^5$ is

(a) 256 (b) 128 (c) 512 (d) 64

32. Let $(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2}+\ldots x, n\in R$. Then the sum of the series $3+\frac{8}{3}+\frac{80}{3^3}+\frac{240}{3^4}+\ldots$ is

(a) 9 (b) 27 (c) 12 (d) 101

33. $13^{99}-19^{93}$ is divisible by 162, then the remainder is

(a) 1 (b) 0 (c) 3 (d) 4

34. The remainder when $6^{83}+8^{83}$ is divided by 49 is equal to

(a) 1 (b) 2 (c) 35 (d) 4

35. If $x+y=1$, then $\sum_{k=0}^{n} k^2\,{}^nC_k\cdot x^k\cdot y^{n-k} =$

(a) $nxy-n^2x^2$ (b) $nxy+n^2x^2$

(c) 0 (d) None of these

36. $C_1-C_3+C_5-\ldots=$

(a) $2^{n/2}\cos\frac{n\pi}{4}$ (b) $2^{n/2}\sin\frac{n\pi}{4}$

(c) 0 (d) None of these

37. Find the value of x for which the fourth term in the expansion

$$\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}}+\frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8 \text{ is } 336.$$

(a) 4 (b) 5

(c) 6 (d) 1

38. Number of terms free from radical sign in the expansion of $(1+3^{1/3}+7^{1/7})^{10}$ is

(a) 4 (b) 5 (c) 6 (d) 8

39. Find last three digits in 19^{100}.

(a) 004 (b) 001

(c) 002 (d) 011

40. If m is a positive integer next above $(\sqrt{3}+1)^{2m}$ contains as a factor

(a) 2^{m+1} (b) 2^m

(c) $2^{(m+1)/2}$ (d) $2^{m/2}$

Parikshit (JEE Advanced Level)

SINGLE CORRECT TYPE QUESTIONS

1. The largest real value for x such that

$$\sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{8}{3} \text{ is}$$

(a) $2\sqrt{2}-5$ (b) $2\sqrt{2}+5$

(c) $-2\sqrt{3}-5$ (d) $-2\sqrt{2}+5$

2. Sum of the series $S=\sum_{r=0}^{n}\frac{3^{r+4}\left({}^nC_r\right)}{{}^{r+4}C_4}+\sum_{r=0}^{3}\frac{{}^{n+4}C_r 3^r}{{}^{n+4}C_4}$ is

(a) 4^{n+4} (b) $4^{n+4}\left({}^{2n}C_4\right)$

(c) $\frac{4^{n+4}}{{}^{n+4}C_4}$ (d) $\frac{3^{n+4}+2^{n+4}}{{}^{n+4}C_4}$

3. Suppose $[x]$ denote the greatest integer ≤, and $n\in N$, then

$$\lim_{n\to\infty}\frac{\left[{}^nC_0x^2\right]+\left[{}^nC_1x^2\right]+\cdots+\left[{}^nC_nx^2\right]}{2^n-2} \text{ is equal to-}$$

(a) $\frac{1}{2}x^2$ (b) x^2

(c) $2x^2$ (d) $4x^2$

4. The remainder on dividing $1234^{567}+89^{1011}$ by 12 is

(a) 1 (b) 5

(c) 9 (d) None of these

5. The value of $\sum_{r=0}^{s}\sum_{\substack{s=1\\ r\le s}}^{n} {}^nC_s\,{}^sC_r$ is

(a) 3^n-1 (b) 3^n+1

(c) 3^n (d) $3(3^n-1)$

6. Let $N = 2^{1224} - 1$, $\alpha = 2^{153} + 2^{77} + 1$ and $\beta = 2^{408} - 2^{204} + 1$. Then which of the following statement is correct ?

(*a*) α divides N but β does not

(*b*) β divides N but α does not

(*c*) α and β both divide N

(*d*) Neither α nor β divides N

7. The remainder when $2^{30}.3^{30}$ is divided by 7 is:

(*a*) 1 (*b*) 2 (*c*) 4 (*d*) 6

8. The coefficient of x^{83} in $(1 + x + x^2 + x^3 + x^4)^n (x - 1)^{n+3}$ is equal to

(*a*) ${}^nC_7(-1)^n$ (*b*) $(-1)^n\,{}^nC_{16}$

(*c*) ${}^nC_{13}$ (*d*) ${}^nC_9$

9. Suppose m and n are positive integers and let

$$S = \sum_{k=0}^{n} (-1)^k \frac{1}{k+m+1} \left({}^nC_k\right) \text{ and}$$

$$T = \sum_{k=0}^{m} (-1)^k \frac{1}{k+n+1} \left({}^mC_k\right)\text{, then } S - T \text{ is equal to}$$

(*a*) 0 (*b*) $n^m - m^n$

(*c*) $(n + 1)^m - (m + 1)^n$ (*d*) $(1 - n)^m - (1 - m)^n$

10. Let $S = \sum_{r=0}^{n} \dfrac{{}^nC_r}{\sum_{s=0}^{r} \dfrac{{}^sC_r}{\sum_{t=0}^{s} {}^sC_t \left(2\sin\frac{\pi}{10}\right)^t}}$

Then $S^{1/n}$ is equal to

(*a*) $2\cos(\pi/5)$ (*b*) $2\cos(\pi/10)$

(*c*) $2\sin(\pi/10)$ (*d*) $2\sin(\pi/5)$

11. $\sum_{r=0}^{300} a_r x^r = (1 + x + x^2 + x^3)^{100}$. If $a = \sum_{r=0}^{300} a_r$ then $\sum_{r=0}^{300} r \cdot a_r$ is equal to

(*a*) 300 *a* (*b*) 100 *a* (*c*) 150 *a* (*d*) 75 *a*

12. Sum of $\sum\limits_{0 \le i < j \le 20}\sum (C_i + C_j)$ is equal to

(*a*) $20 \cdot 2^{19}$ (*b*) $20 \cdot 2^{20}$

(*c*) $20 \cdot 2^{18}$ (*d*) $20 \cdot 2^{21}$

13. $\sum_{r=1}^{10} r^3 . {}^{10}C_r$ is not divisible by

(*a*) 64 (*b*) 81 (*c*) 512 (*d*) 128

14. $C_0^2 + \dfrac{C_1^2}{2} + \dfrac{C_2^2}{3} + \dots\dots + \dfrac{C_n^2}{n+1} =$

(*a*) $\dfrac{(2n+1)!}{\{(n)!\}^2}$ (*b*) $\dfrac{(2n)!}{\{(n+1)!\}^2}$

(*c*) $\dfrac{(2n+1)!}{\{(n+1)!\}^2}$ (*d*) $\dfrac{(2n)!}{\{(n)!\}^2}$

15. ${}^{404}C_4 - {}^4C_1\,{}^{303}C_4 + {}^4C_2\,{}^{202}C_4 - {}^4C_3\,{}^{101}C_4$ is equal to

(*a*) $(401)^4$ (*b*) $(101)^4$

(*c*) 0 (*d*) $(201)^4$

16. If $\left|\sum_{k=0}^{10} \dfrac{{}^{10}C_k\,{}^{10+k}C_k}{(-2)^k}\right| = \dfrac{p}{q}$ where p and q are coprime, then the value of $q - p$ is

(*a*) 128 (*b*) 63 (*c*) –65 (*d*) 65

17. The coefficient of x^{15} in the product of $(1 - x)(1 - 2x)(1 - 2^2x)\dots\dots(1 - 2^{15}x)$ is equal to

(*a*) $2^{105} - 2^{121}$ (*b*) $2^{121} - 2^{105}$

(*c*) $2^{120} - 2^{104}$ (*d*) $-2^{121} - 2^{105}$

18. If ${}^{40}C_1 \cdot x(1 - x)^{39} + 2 \cdot {}^{40}C_2\, x^2 (1 - x)^{38} + 3\, {}^{40}C_3\, x^3 (1 - x)^{37} + \dots\dots\dots + 40 \cdot {}^{40}C_{40}\, x^{40} = ax + b$, then $a + b$ is equal to

(*a*) 40 (*b*) –40 (*c*) 20 (*d*) –20

19. If C_r are the binomial coefficients in the expansion of $(1 + x)^n$, the value of

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (i + j) C_i C_j \text{ is}$$

(*a*) $n \cdot 2^n(2^n + 1)$ (*b*) $n \cdot 2^n(2^n - 1)$

(*c*) $2^n(2^n - 1)$ (*d*) $2^n(2^n + 1)$

20. The value of ${}^nC_0 \sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \dots + {}^nC_n \sin(n + 1)x$ is equal to

(*a*) $2^n \sin^n \dfrac{x}{2} \sin\left(\dfrac{n}{2} + 1\right)x$ (*b*) $2^n \cos^n \dfrac{x}{2} \sin\left(\dfrac{n}{2} + 1\right)x$

(*c*) $2^n \cos^n \dfrac{x}{2} \cos\left(\dfrac{n}{2} + 1\right)x$ (*d*) $2^n \sin^n \dfrac{x}{2} \cos\left(\dfrac{n}{2} + 1\right)x$

21. Which of the following is true for $1 < r < 2n$

(*a*) $(np + pr)a_r = (r + 1)a_{r+1} + (r - 1)a_{r-1}$

(*b*) $(np - pr)a_r = (r + 1)a_{r+1} + (r - 1 - 2n)a_{r-1}$

(*c*) $(np - pr)a_r = (r + 1)a_{r+1} + (r - 1 - n)a_{r-1}$

(*d*) $(2np + pr)a_r = (r + 1 + n)a_{r+1} + (r + 1 - n)a_{r-1}$

22. The remainder obtained when $a_1 + 5a_2 + 9a_3 + 13a_4 + \dots + (8n - 3)a_{2n}$ is divided by $(p + 2)$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 0

23. The value of $a_1 + 3a_2 + 5a_3 + 7a_4 + \dots + (4n - 1)a_{2n}$, when $p = -3$ and $n \in$ even is

(*a*) n (*b*) $2n - 1$ (*c*) $2n - 2$ (*d*) $2n$

24. If $S_n = {}^nC_1 - \left(1 + \dfrac{1}{2}\right){}^nC_2 + \left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right){}^nC_3$

$$+ \dots. + (-1)^{n-1}\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right){}^nC_n$$

then S_n equal to

(*a*) $\dfrac{2^n - 1}{n(n+1)}$ (*b*) $\dfrac{1}{n}$

(*c*) $\dfrac{2}{n(n+1)}$ (*d*) $\dfrac{2^n}{n(n+1)}$

25. If N is a prime number which divides $S = {}^{39}P_{19} + {}^{38}P_{19} + \dots + {}^{20}P_{19}$; then the largest possible value of N among following is

(a) 41 (b) 37 (c) 31 (d) 19

MULTIPLE CORRECT TYPE QUESTIONS

26. Let $S_n = \sum_{r=0}^{n} (-2)^r \left(\frac{{}^nC_r}{{}^{r+2}C_r} \right)$, then

(a) $S_n = \frac{1}{n+1}$ if n is odd (b) $S_n = \frac{1}{n+2}$ if n is odd

(c) $S_n = \frac{1}{n+1}$ if n is even (d) $S_n = \frac{1}{n+2}$ if n is even

27. Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1x + A_2x^2 + \dots$. If A_0, A_1, A_2 are in $A.P.$ then the value of n is

(a) 2 (b) 3 (c) 5 (d) 7

28. If $(9 + \sqrt{80})^n = I + f$, where I, n are positive integers and $0 < f < 1$, then:

(a) I is an odd integer (b) I is an even integer

(c) $(I + f)(1 - f) = 1$ (d) $1 - f = (9 - \sqrt{80})^n$

29. If recursion polynomials $P_k(x)$ are defined as $P_1(x) = (x - 2)^2$, $P_2(x) = ((x - 2)^2 - 2)^2$

$P_3(x) = ((x - 2)^2 - 2)^2 - 2)^2$ ... (In general $P_k(x) = (P_{k-1}(x) - 2)^2$, then the constant term in $P_k(x)$ is

(a) 4 (b) 2

(c) 16 (d) A perfect square

30. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \dots + a_{20}x^{20}$, then:

(a) $a_1 = 20$ (b) $a_2 = 210$

(c) $a_4 = 8085$ (d) $a_{20} = 2^2 . 3^7 . 7$

COMPREHENSION BASED QUESTIONS

Comprehension (No. 31 to 33)

We know that of ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ be binomial coefficients, then $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + C_3x^3 + \dots + C_n x^n$. Various relations among binomial coefficients can be derived by putting $x = 1, -1, i, \omega$, $\left(\text{where } i = \sqrt{-1}, \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)$.

31. The value of ${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \dots$ must be

(a) $2^{n/2} \cos \frac{n\pi}{2}$ (b) $2^{n/2} \sin \frac{n\pi}{2}$

(c) $2^{n/2} \cos \frac{n\pi}{4}$ (d) $2^{n/2} \sin \frac{n\pi}{4}$

32. The value of expression $({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \dots)^2 + ({}^nC_1 - {}^nC_3 + {}^nC_5 \dots)^2$ must be

(a) 2^{2n} (b) 2^n (c) 2^{n^2} (d) $2^{n/2}$

33. The value of ${}^nC_0 + {}^nC_3 + {}^nC_6 + \dots$ must be

(a) $\frac{2^n}{3}$ (b) $\frac{1}{3}\left(2^n + \cos\frac{n\pi}{3}\right)$

(c) $\frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$ (d) $\frac{1}{3}\left(2^n + 2\sin\frac{n\pi}{3}\right)$

MATCH THE COLUMN TYPE QUESTIONS

34. Match the column:

	Column-I		Column-II
A.	The sum $\sum_{0 \le i < j \le n} \sum ({}^nC_i + {}^nC_j)^2$ may be	p.	$\frac{1}{2}[2^{2n} - {}^{2n}C_n]$
B.	The sum $\sum_{0 \le i < j \le n} \sum ({}^nC_i \cdot {}^nC_j)$ may be	q.	$n^2 \cdot 2^{n-1}$
C.	The sum $\frac{1}{n} \sum_{0 \le i < j \le n} \sum (i + j)({}^nC_i \cdot {}^nC_j)$ may be	r.	$\sum_{r=0}^{n-1} {}^{2n}C_r$
D.	The sum $\sum_{0 \le i < j \le n} \sum (i . {}^nC_i \cdot + j \cdot {}^nC_j)$ may be	s.	$(n - 1)^{2n}C_n + 2^{2n}$

(a) A → s, B → p, C → p, D → q

(b) A → r, B → q, C → p, D → s

(c) A → p, B → s, C → r, D → q

(d) A → s, B → q, C → r, D → p

35. Match the columns:

	Column-I		Column-II
A.	If $f(n) = \sum_{r=1}^{n} r \cdot {}^nC_r$ than f(1) =	p.	1
B.	If $f(n) = \sum_{r=1}^{n} r^2 \cdot {}^nC_r$ then $f(2) =$	q.	6
C.	If $f(n) = \sum_{r=1}^{n} (-1)^r r^2 \cdot C_r$ then $f(3)$	r.	0
D.	If $f(n) = \sum_{r=1}^{n} r^3 \cdot {}^nC_r$ then $\frac{f(4)}{32} =$	s.	7

(a) A → p, B → q, C → r, D → s

(b) A → r, B → q, C → p, D → s

(c) A → p, B → s, C → r, D → q

(d) A → s, B → q, C → r, D → p

36. Match the Column:

	Column-I		Column-II
A.	If $(r + 1)^{th}$ term is the first negative term in the expansion of $(1 + x)^{7/2}$, then the value of r (where $\lvert x \rvert < 1$) is	p.	Divisible by 2
B.	The coefficient of y in the expansion of $(y^2 + 1/y)^5$ is	q.	Divisible by 5
C.	${}^nC_r$ is divisible by n, $(1 < r < n)$ if n is	r.	Divisible by 10
D.	The coefficient of x^4 in the expansion of $(1 + 2x + 3x^2 + 4x^3 + \dots \text{ up to } \infty)^{1/2}$ is c, $(c \in N)$, then $c + 1$ (where $\lvert x \rvert < 1$) is	s.	A prime number

(a) A → p, B → q, C → r, D → p, q, r

(b) A → q, s, B → p, q, r, C → s, d, D → p, s

(c) A → p, B → p, q, r, C → r, D → q

(d) A → p, q, r, B → q, C → r, D → p

37. Match the Column:

	Column-I		Column-II
A.	If $x = (7 + 4\sqrt{3})^{2n} = [x] + f$, then $x(1 - f) =$	p.	6
B.	If second, third and fourth terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, then n is equal to	q	1
C.	Value of ${}^4C_0{}^4C_4 - {}^4C_1{}^4C_3 + {}^4C_2{}^4C_2 - {}^4C_3{}^4C_1 + {}^4C_4{}^4C_0$ is	r.	2
D.	If x is very large as compare to y, then the value of k in $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = 1 + \frac{y^2}{kx^2}$	s.	5

(a) A → p, B → q, C → r, D → s

(b) A → r, B → q, C → p, D → s

(c) A → q, B → s, C → p, D → r

(d) A → s, B → q, C → r, D → p

38. Match the columns:

	Column-I		Column-II
A.	The coefficient of two consecutive terms in the expansion of $(1 + x)^n$ will be equal then n can be	p.	9
B.	If $15^n + 23^n$ is divided by 19, then n can be	q.	10
C.	${}^{10}C_0\ {}^{20}C_{10} - {}^{10}C_1\ {}^{18}C_{10} + {}^{10}C_2{}^{16}C_{10}$ -....is divisible by 2^n, then n can be	r.	11
D.	If the coefficients of T_r, T_{r+1}, T_{r+2} terms of $(1 + x)^{14}$ are in A.P., then r is less than	s.	12
		t.	14

(a) A → p, B → q, C → r, D → p, q, r

(b) A → q, s, B → p, q, r, C → s, t, D → p, s

(c) A → p, B → p, q, r, C → r, D → q

(d) A → p, r, B → p, r, C → p, q, D → q, r, s, t

39. Match the columns:

	Column-I		Column-II
A.	$\sum_{i\neq j}\sum {}^{10}C_i\,{}^{10}C_j$	p.	$\frac{2^{20}-{}^{20}C_{10}}{2}$
B.	$\sum_{0\le i\le}\ \sum_{j\le n} {}^{10}C_i\,{}^{10}C_j$	q.	$2^{20}-{}^{20}C_{10}$
C.	$\sum_{0\le i<}\ \sum_{j\le n} {}^{10}C_i\,{}^{10}C_j$	r.	2^{20}
D.	$\sum_{i=0}^{10}\sum_{j=0}^{10} {}^{10}C_i\,{}^{10}C_j$	s.	$\frac{2^{20}+{}^{20}C_{10}}{2}$

(a) A → q, B → s, C → p, D → r

(b) A → s, B → q, C → p, D → r

(c) A → p, B → s, C → q, D → r

(d) A → q, B → r, C → p, D → s

40. Let $N = a_n 10^n + a_{n-1}10^{n-1} + \ldots + a_2 10^2 + a_1 10 + a_0$ where $0 \le a_i \le 9$ and $a_n \neq 0$. Let us define a function $T(N) = (a_n + 1)(a_{n-1} + 1)\ldots(a_2 + 1)(a_1 + 1)(a_0 + 1)$. Again we define the $T^{-1}(X)$ as the smallest N for which $T(N) = X$, then

	Column-I		Column-II
A.	The value of $N < 100$ such that $T(N) = \frac{7N}{8}$ is	p.	441
B.	The value of $T^{-1}(54)$ is	q.	77
C.	The value of X for which $T^{-1}(X)$ does not exist is	r.	58
D.	The value of $T^{-1}(50)$ if $T^{-1}(X)$ is defined as the greatest number for which $T(N) = X$	s.	16

(a) A → s, B → r, C → q, D → p

(b) A → s, B → q, C → p, D → r

(c) A → p, B → s, C → q, D → r

(d) A → q, B → r, C → p, D → s

INTEGER TYPE QUESTIONS

41. The index 'n' of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ if for $x = 1$ the 9th term of the expansion has numerically the greatest term ($n \in N$), is:

42. If $(6\sqrt{6}+14)^{2n+1} = N$, and if F be the fractional part of N, NF is $20^{\lambda n+\mu}$. Find $\lambda + \mu$.

Credit:--- Divansh.....

43. Let $(x^3+px^2+2x-5)^{19}(x^2+qx-41)^8(x^4-x^3+x-7)^6 = x^{97}+391x^{96}+a_{95}x^{95}+a_{94}x^{94}+\dots a_1x+a_0$ be an identity, where $p, q, a_{95} \dots a_0$ are integers. Compute the smallest positive value of p.

44. If $\sum_{r=0}^{n}\frac{C_r\cdot 3^{r+4}}{(r+1)(r+2)(r+3)(r+4)} = \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left[k^{n+k}-\sum_{s=0}^{3}{}^{n+4}C_s\cdot 3^s\right]$ then $k=$

45. If C_r are the binomial co-efficient in the expansion of $(1+x)^n$, the value of $\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)C_iC_j$ is $n\cdot 2^{\lambda n}$. Find λ.

46. $2222^{5555}+5555^{2222}=7k+r$, then $r=$

47. If $\sum_{m=0}^{n}{}^nC_m\cdot\sin(mx)\cdot\cos(n-m)x=k^{n-1}\sin(nx)$, then $k=$

48. Real value for x such that $\sum_{k=0}^{4}\left(\frac{3^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{32}{3}$ is

49. If ${}^nC_0\cdot{}^{2n}C_n-{}^nC_1\cdot{}^{2n-2}C_n+{}^nC_2\cdot{}^{2n-4}C_n+\dots=2^n+\lambda$, find λ

50. Find the number of divisors of the number

$$N={}^{2000}C_1+2\cdot{}^{2000}C_2+3\cdot{}^{2000}C_3+\dots\dots+2000\cdot{}^{2000}C_{2000}$$

51. If $\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r$ and $a_k=1$ for all $k\ge n$, if $b_n={}^{2n+\lambda}C_{n+\mu}$, find $\lambda+\mu$.

52. If then $\sum_{r=0}^{n}\frac{2r}{{}^nC_r}=\sum_{r=0}^{n}\frac{n^2-3n+3}{{}^nC_r}$ then find sum of all the possible values of n.

53. If it is given that coefficient of 2nd, 3rd and 4th term in expansion of $(a+b)^n$ are in A.P. and the value of 3rd term is 84 $(a, b\in(1,\infty))$, then the maximum value of $(\log_{2^{1/3}}a^2)(\log_{2^{1/2}}b^2)$ is equal to ________.

PYQ's (Past Year Questions)

BINOMIAL EXPRESSION, BINOMIAL THEOREM

1. Remainder when $64^{32^{n}}$ is divided by 9 is equal to ____.
[29 Jan, 2024 (Shift-II)]

2. Number of integral terms in the expansion of $\left\{7^{\left(\frac{1}{2}\right)}+11^{\left(\frac{1}{6}\right)}\right\}^{824}$ is equal to______.
[30 Jan, 2024 (Shift-I)]

3. In the expansion of

$(1+x)(1-x^2)\left(1+\frac{3}{x}+\frac{3}{x^2}+\frac{1}{x^3}\right)^5$, $x\ne 0$, the sum of the coefficient of x^3 and x^{-13} is equal to _______.
[31 Jan, 2024 (Shift-I)]

4. Let the coefficient of x^r in the expansion of

$(x+3)^{n-1}+(x+3)^{n-2}(x+2)+(x+3)^{n-3}(x+2)^2+\dots+(x+2)^{n-1}$ be α_r. If $\sum_{r=0}^{n}\alpha_r=\beta^n-\gamma^n, \beta,\gamma\in N$, then the value of $\beta^2+\gamma^2$ equals ______.
[31 Jan, 2024 (Shift-II)]

5. Let m and n be the coefficients of seventh and thirteenth terms respectively in the expansion of $\left(\frac{1}{3}x^{\frac{1}{3}}+\frac{1}{2x^{\frac{2}{3}}}\right)^{18}$. Then $\left(\frac{n}{m}\right)^{\frac{1}{3}}$ is
[01 Feb, 2024 (Shift-II)]

(a) $\frac{4}{9}$ (b) $\frac{1}{9}$ (c) $\frac{1}{4}$ (d) $\frac{9}{4}$

6. If the term independent of x in the expansion of $\left(\sqrt{a}x^2+\frac{1}{2x^3}\right)^{10}$ is 105, then a^2 is equal to:
[08 April, 2024 (Shift-II)]

(a) 4 (b) 9 (c) 6 (d) 2

7. The remainder when 428^{2024} is divided by 21 is ______.
[09 April, 2024 (Shift-I)]

8. If the ratio of the fifth term from the begining to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$, then the third term from the beginning is:
[06 April, 2023 (Shift-I)]

(a) $60\sqrt{2}$ (b) $60\sqrt{3}$ (c) $30\sqrt{2}$ (d) $30\sqrt{3}$

9. $25^{190}-19^{190}-8^{190}+2^{190}$ is divisible by
[08 April, 2023 (Shift-II)]

(a) 34 but not by 14 (b) Both 14 and 34
(c) Neither 14 nor 34 (d) 14 but not by 34

10. The remainder when $19^{200}+23^{200}$ is divided by 49, is ______.
[01 Feb, 2023 (Shift-I)]

11. If the constant term in the binomial expansion of $\left(\frac{x^{\frac{5}{2}}}{2}-\frac{4}{x^l}\right)^9$ is -84 and the Coefficient of x^{-3l} is $2^\alpha\beta$, where $\beta<0$ is an odd number, Then $|\alpha l-\beta|$ is equal to
[31 Jan, 2023 (Shift-II)]

12. If $\sum_{K=1}^{10}K^2({}^{10}C_K)^2=22000L$, then L is equal to ________.
[29 July, 2022 (Shift-II)]

13. If the coefficients of x and x^2 in the expansion of $(1+x)^p(1-x)^q$, $p, q\ge 15$, are -3 and -5 respectively, then the coefficient of x^3 is equal to __________.
[26 July, 2022 (Shift-I)]

14. The remainder when $7^{2022}+3^{2022}$ is divided by 5 is:
[28 July, 2022 (Shift-I)]

(a) 0 (b) 2 (c) 3 (d) 4

15. Let $(1+x+2x^2)^{20}=a_0+a_1x+a_2x^2+\dots+a_{40}x^{40}$ then $a_1+a_3+a_5+\dots+a_{37}$ is equal to: [18 March, 2021 (Shift-I)]

(a) $2^{20}(2^{20}-21)$ (b) $2^{19}(2^{20}-21)$
(c) $2^{19}(2^{20}+21)$ (d) $2^{20}(2^{20}+21)$

16. The total number of two digit numbers 'n', such that 3^n+7^n is a multiple of 10, is ________.
[25 Feb, 2021 (Shift-II)]

17. The term independent of x in the expansion of

$\left[\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right]^{10}$, $x\ne 1$, is equal to ____
[18 March, 2021 (Shift-II)]

18. Let $n\in N$ and $[x]$ denote the greatest integer less than or equal to x. If the sum of $(n+1)$ terms ${}^nC_0, 3.{}^nC_1, 5.{}^nC_2, 7.{}^nC_3, \dots$ is equal to $2^{100}.101$, then $2\left[\frac{n-1}{2}\right]$ is equal to ________.
[25 July, 2021 (Shift-II)]

19. If the coefficient of a^7b^8 in the expansion of $(a+2b+4ab)^{10}$ is $K.2^{16}$, then K is equal to ________.
[31 Aug, 2021 (Shift-II)]

20. If the number of integral terms in the expansion of $(3^{1/2}+5^{1/8})^n$ is exactly 33, then the least value of n is:
[03 Sep, 2020 (Shift-I)]

(a) 264 (b) 128 (c) 256 (d) 248

21. The coefficient of x^7 in the expression $(1+x)^{10}+x(1+x)^9+x^2(1+x)^8+\dots\dots+x^{10}$ is: [07 Jan, 2020 (Shift-II)]

(a) 210 (b) 330 (c) 420 (d) 120

22. The total number of irrational terms in the binomial expansion of $(7^{1/5}-3^{1/10})^{60}$ is: [12 Jan, 2019 (Shift-II)]

(a) 55 (b) 49 (c) 48 (d) 54

23. A ratio of the 5th term from the beginning to the 5th term from the end in the binomial expansion of $\left(2^{\frac{1}{3}}+\frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$ is:
[12 Jan, 2019 (Shift-I)]

(a) $1:2(6)^{\frac{1}{3}}$ (b) $1:4(16)^{\frac{1}{3}}$
(c) $4(36)^{\frac{1}{3}}:1$ (d) $2(36)^{\frac{1}{3}}:1$

Credit:--- Divansh.....

24. Let $(x+10)^{50}+(x-10)^{50}=a_0+a_1x+a_2x^2+\ldots+a_{50}x^{50}$, for $x \in R$; then $\frac{a_2}{a_0}$ is equal to: [11 Jan, 2019 (Shift-II)]

(a) 12.50 (b) 12.00 (c) 12.25 (d) 12.75

BINOMIAL COEFFICIENTS AND THEIR SUMS

25. $^{n-1}C_r=(k^2-8)\,^nC_{r+1}$ if and only if: [27 Jan, 2024 (Shift-I)]

(a) $2\sqrt{2}<k\leq 3$
(b) $2\sqrt{3}<k\leq 3\sqrt{2}$
(c) $2\sqrt{3}<k<3\sqrt{3}$
(d) $2\sqrt{2}<k<2\sqrt{3}$

26. If A denotes the sum of all the coefficients in the expansion of $(1-3x+10x^2)^n$ and B denotes the sum of all the coefficients in the expansion of $(1+x^2)^n$, then: [27 Jan, 2024 (Shift-I)]

(a) $A=B^3$ (b) $3A=B$ (c) $B=A^3$ (d) $A=3B$

27. The coefficient of x^{2012} in the expansion of $(1-x)^{2008}(1+x+x^2)^{2007}$ is equal to [27 Jan, 2024 (Shift-II)]

28. If $\frac{^{11}C_1}{2}+\frac{^{11}C_2}{3}+\ldots+\frac{^{11}C_9}{10}=\frac{n}{m}$ with $gcd(n,m)=1$, then $n+m$ is equal to ______. [29 Jan, 2024 (Shift-I)]

29. Let $\alpha=\sum_{k=0}^{n}\left(\frac{(^nC_k)^2}{k+1}\right)$ and $\beta=\sum_{k=0}^{n-1}\left(\frac{^nC_k\,^nC_{k+1}}{k+2}\right)$.

If $5\alpha=6\beta$, then n equals ______. [30 Jan, 2024 (Shift-II)]

30. If for some m, n; $^6C_m+2(^6C_{m+1})+^6C_{m+2}>\,^8C_3$ and $^{n-1}P_3:\,^nP_4=1:8$, then $^nP_{m+1}+\,^{n+1}C_m$ is equal to [31 Jan, 2024 (Shift-II)]

(a) 380 (b) 376 (c) 384 (d) 372

31. If the Coefficient of x^{30} in the expansion of $\left(1+\frac{1}{x}\right)^6(1+x^2)^7(1-x^3)^8$; $x\neq 0$ is α, then $|\alpha|$ equals ________. [01 Feb, 2024 (Shift-I)]

32. The sum of all rational terms in the expansion of $\left(2^{\frac{1}{5}}+5^{\frac{1}{3}}\right)^{15}$ is equal to: [04 April, 2024 (Shift-I)]

(a) 3133
(b) 633
(c) 931
(d) 6131

33. Let $a=1+\frac{^2C_2}{3!}+\frac{^3C_2}{4!}+\frac{^4C_2}{5!}+\ldots$,

$$b=1+\frac{^1C_0+\,^1C_1}{1!}+\frac{^2C_0+\,^2C_1+\,^2C_2}{2!}+\frac{^3C_0+\,^3C_1+\,^3C_2+\,^3C_3}{3!}+\ldots$$

Then $\frac{2b}{a^2}$ is equal to ______ [04 April, 2024 (Shift-I)]

34. If the coefficients of x^4, x^5 and x^6 in the expansion of $(1+x)^n$ are in the arithmetic progression, then the maximum value of n is: [04 April, 2024 (Shift-II)]

(a) 14 (b) 21 (c) 28 (d) 7

Credit:--- Divansh.....

35. If the constant term is expansion of $(1+2x-3x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ is p, then $108\,p$ is equal to **[05 April, 2024 (Shift-I)]**

36. If the constant term in the expansion of $\left(\frac{\sqrt[4]{3}}{x}+\frac{2x}{\sqrt[3]{5}}\right)^{12}$, $x \neq 0$, is $\alpha \times 2^8 \times \sqrt[4]{3}$, then 25α is equal to: **[05 April, 2024 (Shift-II)]**
(*a*) 639 (*b*) 724 (*c*) 693 (*d*) 742

37. If the second, third and fourth terms in the expansion of $(x+y)^n$ are 135,30 and $\frac{10}{3}$, respectively, then $6(n^3+x^2+y)$ is equal to ________. **[06 April, 2024 (Shift-I)]**

38. Let $0 \leq r \leq n$. If ${}^{n+1}C_{r+1} : {}^{n}C_{r} : {}^{n-1}C_{r-1} = 55 : 35 : 21$, then $2n + 5r$ is equal to: **[06 April, 2024 (Shift-II)]**
(*a*) 60 (*b*) 62
(*c*) 50 (*d*) 55

39. Let $\alpha = \sum_{r=0}^{n}\left(4r^2+2r+1\right){}^{n}C_r$ and $\beta = \left(\sum_{r=0}^{n}\frac{{}^{n}C_r}{r+1}\right)+\frac{1}{n+1}$. If $140 < \frac{2\alpha}{\beta} < 281$, then the value of n is ________. **[08 April, 2024 (Shift-I)]**

40. The coefficient of x^{70} in $x^2(1+x)^{98}+x^3(1+x)^{97}+x^4(1+x)^{96}+\ldots\ldots+x^{54}(1+x)^{46}$ is ${}^{99}C_p - {}^{46}C_q$.
Then a possible value to $p+q$ is: **[09 April, 2024 (Shift-I)]**
(*a*) 55 (*b*) 61 (*c*) 68 (*d*) 83

41. The sum of the coefficient of $x^{2/3}$ and $x^{-2/5}$ in the binomial expansion of $\left(x^{2/3}+\frac{1}{2}x^{-2/5}\right)^9$ is: **[09 April, 2024 (Shift-II)]**
(*a*) $\frac{21}{4}$ (*b*) $\frac{69}{16}$
(*c*) $\frac{63}{16}$ (*d*) $\frac{19}{4}$

42. The coefficient of x^{301} in $(1+x)^{500}+x(1+x)^{499}+x^2(1+x)^{498}+\ldots+x^{500}$ is: **[30 Jan, 2023 (Shift-I)]**
(*a*) ${}^{501}C_{302}$ (*b*) ${}^{500}C_{301}$ (*c*) ${}^{500}C_{300}$ (*d*) ${}^{501}C_{200}$

43. If $\sum_{k=1}^{31}({}^{31}C_k)({}^{31}C_{k-1})-\sum_{k=1}^{30}({}^{30}C_k)({}^{30}C_{k-1}) = \frac{\alpha(60!)}{(30!)(31!)}$, where $\alpha \in R$, then the value of 16α is equal to **[28 June, 2022 (Shift-I)]**
(*a*) 1411 (*b*) 1320 (*c*) 1615 (*d*) 1855

44. The coefficient of x^{101} in the expression $(5+x)^{500}+x(5+x)^{499}+x^2(5+x)^{498}+\ldots+x^{500}$, $x>0$, is **[25 June, 2022 (Shift-II)]**
(*a*) ${}^{501}C_{101}(5)^{399}$ (*b*) ${}^{501}C_{101}(5)^{400}$
(*c*) ${}^{501}C_{100}(5)^{400}$ (*d*) ${}^{500}C_{101}(5)^{309}$

45. If the sum of the co-efficients of all the positive even powers of x in the binomial expansion of $\left(2x^3+\frac{3}{x}\right)^{10}$ is $5^{10}-\beta.\,3^9$, then β is equal to......... **[25 June, 2022 (Shift-II)]**

46. The maximum value of the term independent of 't' in the expansion of $\left(tx^{\frac{1}{5}}+\frac{(1-x)^{\frac{1}{10}}}{t}\right)^{10}$ where $x \in (0, 1)$ is: **[26 Feb, 2021 (Shift-I)]**
(*a*) $\frac{10!}{\sqrt{3}(5!)^2}$ (*b*) $\frac{2.10!}{3(5!)^2}$
(*c*) $\frac{10!}{3(5!)^2}$ (*d*) $\frac{2.10!}{3\sqrt{3}(5!)^2}$

47. For integers n and r, let $\binom{n}{r} = \begin{cases} {}^{n}C_r, & \text{if } n \geq r \geq 0 \\ 0, & \text{otherwise} \end{cases}$
The maximum value of k for which the sum $\sum_{i=0}^{k}\binom{10}{i}\binom{15}{k-i}+\sum_{i=0}^{k+1}\binom{12}{i}\binom{13}{k+1-i}$ exists, is equal to **[24 Feb, 2021 (Shift-II)]**

48. Let n be a positive integer. Let
$$A = \sum_{k=0}^{n}(-1)^k\,{}^{n}C_k\left[\left(\frac{1}{2}\right)^k+\left(\frac{3}{4}\right)^k+\left(\frac{7}{8}\right)^k+\left(\frac{15}{16}\right)^k+\left(\frac{31}{32}\right)^k\right]$$
If $63A = 1-\frac{1}{2^{30}}$, then n is equal to ______ **[16 March, 2021 (Shift-II)]**

49. If the sum of the coefficients of all even powers of x in the product $(1+x+x^2+\ldots+x^{2n})(1-x+x^2-x^3+\ldots+x^{2n})$ is 61, then n is equal to ________. **[07 Jan, 2020 (Shift-I)]**

50. For non-negative integers s and r,
let $\binom{s}{r} = \begin{cases} \frac{s!}{r!(s-r)!} & \text{if } r \leq s, \\ 0 & \text{if } r > s. \end{cases}$. For positive integers m and n,
let $(m,n)\sum_{p=0}^{m+n}\frac{f(m,n,p)}{\binom{n+p}{p}}$
where for any nonnegative integer p,
$$f(m,n,p) = \sum_{i=0}^{p}\binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}$$
Then which of the following statements is/are TRUE? **[JEE Adv, 2020]**
(*a*) $g(m, n) = g(n, m)$ for all positive integers m, n
(*b*) $g(m, n+1) = g(m+1, n)$ for all positive integers m, n
(*c*) $g(2m, 2n) = 2g(m, n)$ for all positive integers m, n
(*d*) $g(2m, 2n) = (g(m, n))^2$ for all positive integers m, n

51. If $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}\right)^3 = \frac{k}{21}$, then k equals: **[10 Jan, 2019 (Shift-I)]**
(*a*) 400 (*b*) 50
(*c*) 200 (*d*) 100

Credit:--- Divansh.....

52. If $\sum_{r=0}^{25}\left\{{}^{50}C_r \cdot {}^{50-r}C_{25-r}\right\} = K\left({}^{50}C_{25}\right)$, then K is equal to

[10 Jan, 2019 (Shift-II)]

(*a*) $(25)^2$ (*b*) $2^{25}-1$ (*c*) 2^{24} (*d*) 2^{25}

53. $2.{}^{20}C_0 + 5.{}^{20}C_1 + 8.{}^{20}C_2 + 11.{}^{20}C_3 + ... + 62.{}^{20}C_{20}$ is equal to

[08 April, 2019 (Shift-I)]

(*a*) 2^{23} (*b*) 2^{26} (*c*) 2^{24} (*d*) 2^{25}

54. Suppose $\det\begin{bmatrix}\sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k\end{bmatrix} = 0$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals

[JEE Adv, 2019]

PW Challengers

1. Suppose m and n are positive integers and let

$$S = \sum_{k=0}^{n}(-1)^k \frac{1}{k+m+1}\left({}^nC_k\right) \text{ and}$$

$$T = \sum_{k=0}^{m}(-1)^{k+2} \frac{(k+1)}{(m+1)(k+n+1)}\left({}^{m+1}C_{k+1}\right),$$

then $S - T$ is equal to

(*a*) 0 (*b*) $n^m - m^n$

(*c*) $(n+1)^m - (m+1)^n$ (*d*) $(1-n)^m - (1-m)^n$

2. Let $S = \sum_{r=0}^{n} \dfrac{\dfrac{{}^nC_r}{\sum_{i=0}^{r} {}^rC_i}}{\sum_{i=0}^{r} {}^rC_i\left(2\sin\frac{\pi}{10}\right)^i}$

Then $S^{1/n}$ is equal to

(*a*) $2\cos(\pi/5)$ (*b*) $2\cos(\pi/10)$

(*c*) $2\sin(\pi/10)$ (*d*) $2\sin(\pi/5)$

3. For $n > 3$, $a, b \in R$, let $S(n, a, b)$

$$= \sum_{r=0}^{n}(-1)^r \frac{n(n-1)\cdots(n-r+1)}{r!}(a-r)(b-r)$$

then $S\left(2025, \frac{3}{2}, \frac{1}{2}\right)$

4. If $\dfrac{{}^nC_r + 4\,{}^nC_{r+1} + 6\,{}^nC_{r+2} + 4\,{}^nC_{r+3} + {}^nC_{r+4}}{{}^nC_r + 3\,{}^nC_{r+1} + 3\,{}^nC_{r+2} + {}^nC_{r+3}}$

$= \dfrac{n+k}{r+k}$, then the value of k is

5. Consider $(1+x)^{2n} + (1+2x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, $n \in N$. If $\sum_{r=0}^{2n} a_r = f(n)$ then

(*a*) $\sum_{n=1}^{\infty} \frac{1}{f(n)} = \frac{1}{6}$

(*b*) $\sum_{n=1}^{\infty} \frac{1}{f(n)} = \frac{3}{8}$

(*c*) largest value of p for which $f(5)$ is divisible by 2^p is 11.

(*d*) largest value of p for which $f(5)$ is divisible by 2^p is 9.

6. $\sum_{r=0}^{n}(-1)^r\,{}^nC_r \frac{1+r\log_e 10}{(1+\log_e 10^n)^r}$ equals

(*a*) 7 (*b*) -1 (*c*) n (*d*) 0

7. Value of $\dfrac{\sum_{k=0}^{r} {}^nC_k\,{}^{n-2k}C_{r-k}}{\sum_{k=r}^{n} {}^nC_k\,{}^{2k}C_{2r}\left(\frac{3}{4}\right)^{n-k}\left(\frac{1}{2}\right)^{2k-2r}}(n \geq 2r)$ is

(*a*) 2 (*b*) 9 (*c*) 1 (*d*) 0

8. The sum of the series $\sum_{r=1}^{3n-1}\frac{(-1)^{r-1} r}{{}^{3n}C_r}$ is (where n is an even natural number)

(*a*) 0 (*b*) $\frac{3n}{3n+1}$ (*c*) $\frac{3n+1}{3n+2}$ (*d*) $\frac{3n}{3n+2}$

9. If $\dfrac{{}^nC_1 + 3^2\,{}^nC_3 + 5^2\,{}^nC_5 + ...}{n(n+1)} = 2^{n-\alpha}\ \forall\ n \geq 3, n \in N$ then α is equal to_____

10. If sum of elements of all pairs (m, n) of positive integers satisfying the equation $(2^n - 1)(2^n - 2)(2^n - 4)......(2^n - 2^{n-1}) = m!$ is K. Then find ${}^KC_1$

Credit:--- Divansh.....

ANSWER KEY

CONCEPT APPLICATION

1. (*b*) 2. (*b*) 3. (*b*) 4. (*c*) 5. (*a*) 6. (*b*) 7. (*i*) ${}^{81}C_{21}$ (*ii*) ${}^{100}C_{11}$ (*iii*) ${}^{101}C_{11} - {}^{20}C_{9}$
8. $3^n + 4^n + 5^n$ 9. (*a*) 10. (*d*) 11. (*c*) 12. (*b*) 13. (*a*) 14. (*c*) 15. (*d*) 16. (*d*)
17. (*b*) 18. (*b*) 19. (*d*) 20. (*c*) 21. (*d*)
22. (*i*) 2^{2n-1} (*ii*) $2^{98} - 2^{49}$ (*iii*) $\dfrac{2^{100} - {}^{100}C_{50}}{2}$ (*iv*) 2^{98}
23. (*a,c,d*) 24. (*c*) 25. (*a*) 26. (*b*) 27. (*d*) 28. (*a*) 29. (*a*) 30. (*a*) 31. (*b*) 32. (*c*)

SCHOOL LEVEL PROBLEMS

1. (*c*) 2. (*a*) 3. (*d*) 4. (*a*) 5. (*b*) 6. (*c*) 7. (*c*) 8. (*a*) 9. (*a*) 22. (*b*)
23. (*d*) 24. (*b*) 25. (*b*)

PRARAMBH (TOPICWISE)

1. (*b*) 2. (*c*) 3. (*c*) 4. (*c*) 5. (*b*) 6. (*b*) 7. (*a*) 8. (*c*) 9. (*c*) 10. (*b*)
11. (*d*) 12. (*b*) 13. (*a*) 14. (*a*) 15. (*a*) 16. (*a*) 17. (*a*) 18. (*c*) 19. (*b*) 20. (*c*)
21. (*b*) 22. (*a*) 23. (*c*) 24. (*c*) 25. (*c*) 26. (*c*) 27. (*c*) 28. (*c*) 29. (*a*) 30. (*d*)
31. (*a*) 32. (*a*) 33. (*b*) 34. (*d*) 35. (*b*) 36. (*c*) 37. (*a*) 38. (*a*) 39. (*b*) 40. (*c*)
41. (*a*) 42. (*c*) 43. (*b*)

RABAL (JEE MAIN LEVEL)

1. (*c*) 2. (*a*) 3. (*c*) 4. (*b*) 5. (*c*) 6. (*b*) 7. (*b*) 8. (*a*) 9. (*a*) 10. (*d*)
1. (*a*) 12. (*b*) 13. (*b*) 14. (*a*) 15. (*c*) 16. (*b*) 17. (*c*) 18. (*b*) 19. (*d*) 20. (*d*)
21. (*b*) 22. (*d*) 23. (*b*) 24. (*b*) 25. (*b*) 26. (*d*) 27. (*b*) 28. (*b*) 29. (*c*) 30. (*b*)
31. (*c*) 32. (*b*) 33. (*b*) 34. (*c*) 35. (*b*) 36. (*b*) 37. (*b*) 38. (*c*) 39. (*b*) 40. (*a*)

PARIKSHIT (JEE ADVANCED LEVEL)

1. (*a*) 2. (*c*) 3. (*b*) 4. (*c*) 5. (*a*) 6. (*b*) 7. (*a*) 8. (*b*) 9. (*a*) 10. (*a*)
11. (*c*) 12. (*b*) 13. (*c*) 14. (*c*) 15. (*b*) 16. (*d*) 17. (*a*) 18. (*a*) 19. (*b*) 20. (*b*)
21. (*b*) 22. (*c*) 23. (*d*) 24. (*b*) 25. (*a*) 26. (*b,c*) 27. (*a,b*) 28. (*a,c,d*) 29. (*a, d*) 30. (*a,b,c*)
31. (*c*) 32. (*b*) 33. (*c*) 34. (*a*) 35. (*a*) 36. (*b*) 37. (*c*) 38. (*d*) 39. (*a*) 40. (*a*)
41. [12] 42. [3] 43. [7] 44. [4] 45. [2] 46. [0] 47. [2] 48. [1] 49. [0] 50. [8016]
51. [2] 52. [4] 53. [4]

PYQ's (PAST YEAR QUESTIONS)

1. [1] 2. [138] 3. [118] 4. [25] 5. (*d*) 6. (*a*) 7. [1] 8. (*b*) 9. (*a*) 10. [29]
11. [98] 12. [221] 13. [23] 14. (*c*) 15. (*b*) 16. [45] 17. [210] 18. [98] 19. [315] 20. (*c*)
21. (*b*) 22. (*d*) 23. (*c*) 24. (*c*) 25. (*a*) 26. (*a*) 27. [0] 28. [2041] 29. [10] 30. (*d*)
31. [678] 32. (*a*) 33. [8] 34. (*a*) 35. [54] 36. (*c*) 37. [806] 38. (*c*) 39. [5] 40. (*d*)
41. (*a*) 42. (*d*) 43. (*a*) 44. (*a*) 45. [83] 46. (*d*) 47. [12] 48. [6] 49. [30] 50. (*a, b, d*)
51. (*d*) 52. (*d*) 53. (*d*) 54. [6.20]

PW CHALLENGERS

1. (*a*) 2. (*a*) 3. [0] 4. [4] 5. (*a, c*) 6. (*d*) 7. (*c*) 8. (*d*) 9. [3] 10. [7]

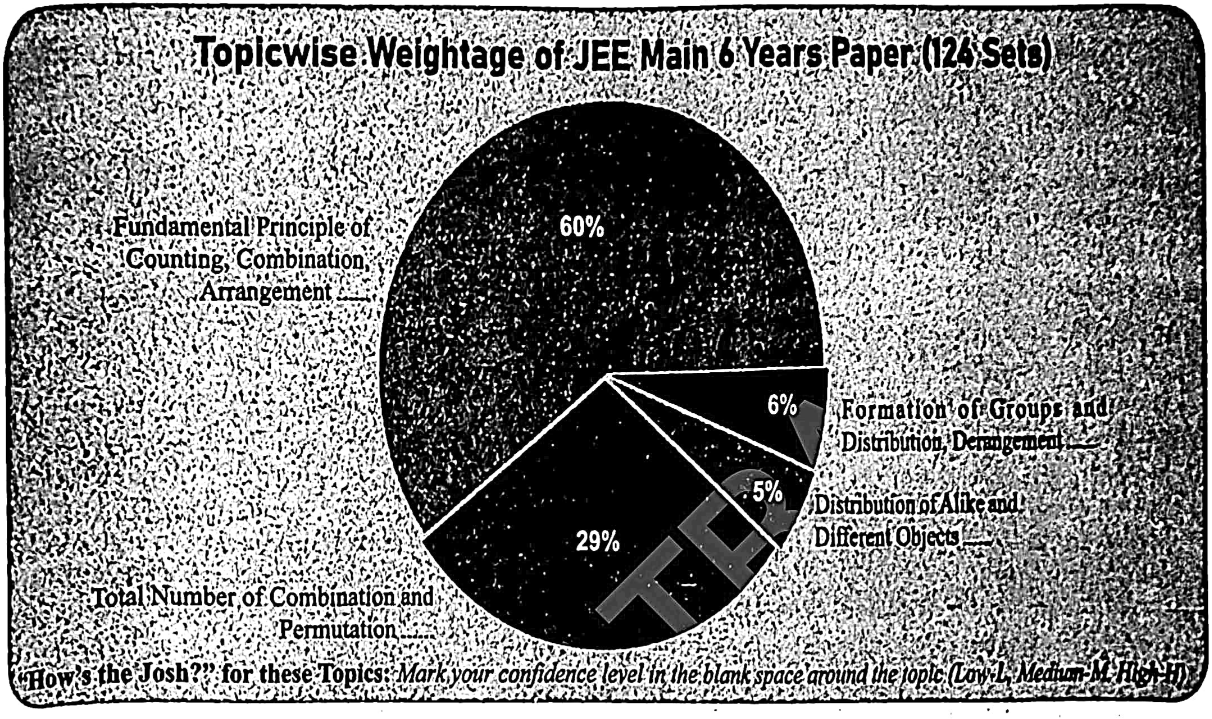
Credit:--- Divansh.....

CHAPTER

8

Permutations and Combinations

$^nP_r = \frac{n!}{(n-r)!}$



"How's the Josh?" for these Topics: *Mark your confidence level in the blank space around the topic (Low-L, Medium-M, High-H)*

INTRODUCTION

A branch of mathematics where we count number of objects or number of ways of doing a particular job without actually counting them, is known as combinatorics and in this chapter we will deal with elementary combinatorics.

For example, if in a room there are five rows of chairs and each rows contains seven chairs, then without counting them we can say, total number of chairs is 35.

NOTATION OF FACTORIAL

Notation of Factorial and its Algebra

The continued product of first n, natural numbers is called as "n factorial" and dentoed by $n!$ or $\underline{|n}$

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots (n-1)n$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$3! = 1 \cdot 2 \cdot 3 = 6$

$5! = 120;\ 6! = 720;\ 7! = 5040$

Special Results

- $0! = 1$ i.e. factorial of zero is 1

  Proof: $n! = (n-1)! \cdot n$

  Putting $n = 1$

  $1! = (1-1)!\ 1 \Rightarrow 0! = 1$
- Factorial of negative number is undefined

Illustrating the concepts:

(*i*) Find n if $(n+1)! = 12 \times (n-1)!$

(*ii*) $(n+2)! = 2550n!$

Sol.

(*i*) $(n+1)\,n(n-1)! = 12 \times (n-1)!$

$n^2 + n - 12 = 0;$

$(n+4)(n-3) = 0 \therefore n = 3, (n \neq -4)$

(*ii*) $(n+2)(n+1) = 2550;$

$(n+52)(n-49) = 0 \therefore n = 49, (n \neq -52)$

- $(2n!) = 2^n \cdot n!\,[1 \cdot 3 \cdot 5 \ldots (2n-1)]$

  Proof: $(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots (2n-1)(2n)$

  Take 2 common from each n even terms

  $= 2^n (1 \cdot 3 \cdot 5 \ldots (2n-1))(1 \cdot 2 \cdot 3 \ldots n)$

  $= 2^n (n!)(1 \cdot 3 \cdot 5 \ldots (2n-1))$

FUNDAMENTAL PRINCIPLE OF COUNTING

Addition Principle

If a work can be done in m different ways and another work which is independent of first can be done in n different ways, then either of the two operations can be preformed in $(m+n)$ ways.

Illustrating the Concepts

There are 15 gates to enter a city from north and 10 gates to enter the city from east. In how many ways a person can enter the city?

Number of ways to enter the city from north = 15

Number of ways to enter the city from east = 10

Credit:--- Divansh.....

A person can enter the city from north or from east.

Hence, the number of ways to enter the city = 15 + 10 = 25

Multiplication Principle

If one operation (I) can be done in m different ways and another Operation (II) can be performed in n different ways, then total number of ways in which both of these can be performed together is $m \times n$. If there are more than two operations to be done, then the total number of different ways to do all of them together will be $m \times n \times p \times \ldots$

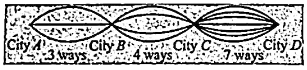
For example: Operation A can be performed in "a, b, c" different ways and operation B can be perfomred in p, q different ways, then possible ways to perform operation A and operation B together are a-p, a-q, b-p, b-q, c-p, c-q i.e., each way of operation A is combined by 2 ways operation B.

i.e., Total ways = 2 + 2 + 2 = 2 added 3 times = 2 × 3 = 6.

In general, total ways = $m \times n$ where m and n are individual ways of performing the operations.

Illustrating the Concepts

There are 3 routes to travel from city A to city B and 4 routes to travel from city B to city C and 7 routes from C to D. In how many different ways (routes) a man can travel from city A to city D via city B and city C.



The man can perform the task of travelling from city A to city B in ways = 3.

The man can perform task of travelling from city B to city C in ways = 4.

Similarly from city C to city D in ways = 7.

Using fundamental principle of counting,

Total routes to travel from A to D via B and via $C = m \times n \times p$ $= 3 \times 4 \times 7 = 84$ routes.

Train Your Brain

Example 1: There are 20 boys and 10 girls in a group. In how many ways can a group leader be selected if the group leader can either be a boy or a girl.

Sol. If it is clear by above sum rule that to select leader, we have 20 boys or 10 girls

⇒ Total number of ways to do the same is 20 + 10 = 30 ways.

Example 2: How many possible out comes are there if a coin is tossed 5 times?

Sol. Each time there are two possible outcomes

∴ Number of outcomes when coin is tossed 5 times = 2 × 2 × 2 × 2 × 2 = 32

Example 3: In how many ways 2 different prizes be awarded to 15 students, without giving both to the same student?

Sol. First prize can be awarded in 15 ways and corresponding to each way of doing so, the second prize can be awarded in 14 ways. Hence, total ways = 15 × 14.

Concept Application

1. There are nine students (5 boys and 4 girls) in the class. In how many ways
   - (*i*) One student (either girl or boy) can be selected to represent the class.
   - (*ii*) A team of two students (one girl and one boy) can be selected.
   - (*iii*) Two medals can be distributed. (No one get both)
   - (*iv*) One prize for maths, two prizes for physics and three prizes for chemistry can be distributed.

     (No student can get more than one prize in same subject and prizes are distinct)

PERMUTATIONS

Each of different arrangement which can be made by taking some or all of a number of things is called a permutation. It is assumed that

- The given things let us say there are n of them are all distinct, that is, no two are alike.
- The arrangement is placing one thing next to another as in a straight line; hence it is also known as linear permutation.
- In any arrangement, any one thing is used only once. In other words, there is no repetition.

Counting Formulae for Permutation

The meaning of ${}^nP_r$

Suppose there are r blank spaces in a row and n different letters. The number of ways of filling up the blank spaces with n different letters is the number of ways of arranging n things r at a time, i.e. ${}^nP_r$. It must be noted that each space has to be filled up with only one letter.

The first space can be filled in n ways. Having filled it, there are $n-1$ letters left and therefore the second space can be filled in $n-1$ ways. Hence the first two spaces can be filled in $n(n-1)$ ways. When the first two spaces are filled, there are $n-2$ letters left, so that the third space can be filled in $n-2$ ways. Therefore the first three spaces can be filled in $n(n-1)(n-2)$ ways; proceeding like this, the r spaces can be filled in $n(n-1)(n-2) \ldots [n-(r-1)]$ ways.

The number of permutations of n things taken r at a time is denoted as ${}^nP_r$ and its value is equal to

$${}^nP_r = n(n-1)(n-2) \ldots (n-r+1)$$

$$= \frac{n!}{(n-r)!} \text{ (using factorial notation)}$$

$n! = n(n-1) \ldots 3.2.1.$ where $0 \le r \le n$.

In particular

- The number of permutations of n different things taken all at a time $= {}^nP_n = n!$

  ${}^nP_0 = 1$, ${}^nP_1 = $ n and ${}^nP_{n-1} = {}^nP_n = n!$

  ${}^nP_r = n({}^{n-1}P_{r-1})$ where $r = 1, 2, \ldots n$.

Permutation for Different Things

Number of permutations under certain conditions:

- Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement is $r.{}^{(n-1)}P_{r-1}$.
- Number of permutations of n different things, taken r at a time, when a particular thing is never taken in any arrangement is ${}^{n-1}P_r$.
- Number of permutations of n different things, taken r at a time, when m particular things are never taken in any arrangement is ${}^{n-m}P_r$.
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $(m!)(n-m+1)!$.
- Number of permutations of n different things, taken all, at a time, when m specified things never come together is $n! - m!(n-m+1)!$

PERMUTATION OF N DISTINCT OBJECT WHEN REPETITION IS ALLOWED

The number of permutations of n different things taken r at time when each thing may be repeated any number of times is n^r.

In other words if a job can be completed in n ways and it is to be repeated r number of times then the total number of ways is n^r.

Also if n distinct things are arranged at r places when repetition is allowed, then the number of arrangements are $\underbrace{n \times n \times \ldots \times n}_{r\text{ times}} = n^r$.

Train Your Brain

Example 4: In a steamer there are stalls for 12 animals, and there are cows, horses, and calves (not less than 12 of each) ready to be shipped; in how many ways can the shipload be made?

Sol. The first stall can be occupied by a cow, a horse or a calf, i.e., it can be occupied in 3 ways. The second stall can also be occupied in 3 ways. Therefore the first 2 stalls can be occupied in 3^2 ways. Proceeding like this, it is clear that the 12 stalls can be occupied in 3^{12} ways, i.e., in 531441 ways.

Example 5: Show that ${}^nP_r = {}^{(n-1)}P_r + r.{}^{(n-1)}P_{(r-1)}$

Sol. ${}^{(n-1)}P_r + r.{}^{(n-1)}P_{(r-1)} = \frac{(n-1)!}{(n-1-r)!} + \frac{r(n-1)!}{(n-r)!}$

$$= (n-1)!\left[\frac{1}{(n-1-r)!} + \frac{r}{(n-r)!}\right] = (n-1)!\,\frac{(n-r+r)}{(n-r)!}$$

$(\because (n-r)! = (n-r)(n-r-1)!)$

$$= \frac{n(n-1)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

A common sense interpretation of the identity above is possible. The number of permutations of r things which may be made from n things, $r.{}^{(n-1)}P_{r-1}$ contain one specified thing and ${}^{(n-1)}P_r$ do not contain that specified thing and these two together give ${}^nP_r$.

Example 6: How many three digit numbers can be formed using the digits 1, 2, 3, 4, 5, without repetition of digits? How many of these are even?

Sol. Three places are to be filled with 5 different objects.

$\therefore$ Number of ways $= {}^5P_3 = 5 \times 4 \times 3 = 60$

For the 2nd part, unit digit can be filled in two ways & the remaining two digits can be filled in ${}^4P_2$ ways.

$\therefore$ Number of even numbers $= 2 \times {}^4P_2 = 24$.

Example 7: If all the letters of the word 'QUEST' are arranged in all possible ways and put in dictionary order, then find the rank of the word QUEST.

Sol. Number of words beginning with $E = {}^4P_4 = 24$

Number of words beginning with $QE = {}^3P_3 = 6$

Number of words beginning with $QS = 6$

Number of words beginning with $QT = 6$.

Next word is 'QUEST'

$\therefore$ its rank is $24 + 6 + 6 + 6 + 1 = 43$.

Example 8: There are m men and n monkeys $(n > m)$. If a man can have any number of monkeys, in how many ways may every monkey have a master?

Sol. The first monkey can select his master by m ways, and after that the second monkey can select his master again by m ways, so can the third. And so on, hence all monkeys can select master by $= m \times m \times m$ ... up to n times $= (m)^n$ ways.

Example 9: How many numbers between 10 and 10,000 can be formed by using the digits 1, 2, 3, 4, 5 if

(*i*) No digit is repeated in any number.

(*ii*) Digits can be repeated.

Sol.

(*i*) Number of two digit numbers $= 5 \times 4 = 20$

Number of three digit numbers $= 5 \times 4 \times 3 = 60$

Number of four digit numbers $= 5 \times 4 \times 3 \times 2 = 120$

Total $= 200$

(*ii*) Number of two digit numbers $= 5 \times 5 = 25$

Number of three digit numbers $= 5 \times 5 \times 5 = 125$

Number of four digit numbers $= 5 \times 5 \times 5 \times 5 = 625$

Total $= 775$

Concept Application

2. If ${}^{(m+n)}P_2 = 56$ and ${}^{m-n}P_2 = 12$ then (m, n) equals

(*a*) (5, 1) (*b*) (6, 2) (*c*) (7, 3) (*d*) (9, 6)

3. How many numbers of four digits greater than 2300 can be formed with the digits 0, 1, 2, 3, 4, 5 and 6; no digit being repeated in any number?

(*a*) 560 (*b*) 590 (*c*) 90 (*d*) 360

4. The letters of the word SURITI are written in all possible orders and these words are written out as in a dictionary.

Then the rank of the words SURITI, is

(*a*) 236 (*b*) 263 (*c*) 326 (*d*) 260

Credit:--- Divansh.....

5. There are $(n+1)$ white and $(n+1)$ black balls, each set numbered 1 to $(n+1)$. The number of ways in which the balls can be arranged in row so that the adjacent balls are of different colours is-

(*a*) $(2n+2)!$ (*b*) $(2n+2)! \times 2$

(*c*) $(n+1)! \times 2$ (*d*) $2\{(n+1)!\}^2$

6. The total number of words which can be formed using letters of the word 'FAILURE' such that consonants always occupy odd places, is-

(*a*) 144 (*b*) 576 (*c*) 5040 (*d*) 625

7. In how many ways four friends can put up in 8 hotels of a town if

(*i*) There is no restriction

(*a*) 4096 (*b*) 4609

(*c*) 4926 (*d*) 4962

(*ii*) No two friends can stay together

(*a*) 1680 (*b*) 1608

(*c*) 1660 (*d*) 1672

(*iii*) All the friends do not stay in same hotel?

(*a*) 4088 (*b*) 4808

(*c*) 4880 (*d*) 4888

Arrangement of n Things When All are Not Distinct

- If given n things are not all distinct, then it is possible that few many be of one kind, and few others may be of second kind, etc. In such case, the number of permutations of n things taken all at a time, where p are alike of one kind, q are alike of second kind and r are alike of third kind and the rest $n-(p+q+r)$ are all distinct is given by

$$\frac{n!}{p!q!r!} \qquad (p+q+r \le n)$$

Train Your Brain

Example 10: Find total number of word's formed by using all letters of the word "IITJEE"

Sol. Ways are $= \dfrac{6!}{2!(2!)}$

Example 11: Consider word ASSASSINATION, find number of ways of arranging the letters of word using all.

Sol. ASSASSINATION contains four S, three A, two N and two I.

$$\text{Total ways} = \frac{13!}{(4!)(3!)(2!)(2!)}$$

Example 12: In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative positions of vowels & consonants?

Sol. The consonants in their positions can be arranged in $\dfrac{4!}{2!} = 12$ ways.

The vowels in their positions can be arranged in $\dfrac{3!}{2!} = 3$ ways.

$\therefore$ Total number of arrangements $= 12 \times 3 = 36$

Concept Application

8. In how many ways can 21 identical white and 19 identical black balls be put in a row so that no two black balls are together?

(*a*) 1470 (*b*) 1540 (*c*) 735 (*d*) 745

9. If out of 8 flags, 5 flags are white (identical) and 3 flags are of red colour (identical), then how many signals can be given by using all of them at a time?

(*a*) 15 (*b*) 28 (*c*) 56 (*d*) 126

10. How many six letter words be made out of the letters of 'ASSIST? In how many words the alphabet S alternates with other letters?

(*a*) 120, 6 (*b*) 720, 12 (*c*) 120, 12 (*d*) 720, 24

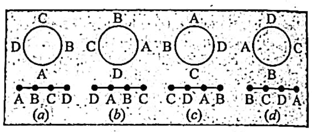
11. In how many ways the alphabets of the word 'MULTIPLE can be ordered other than itself, when the order of vowels is not changed?

(*a*) 3360 (*b*) 3359 (*c*) 6720 (*d*) 20160

CIRCULAR PERMUTATIONS

In the event of the given n things arranged in a circular or even elliptical permutation the first and the last thing in the arrangement are indistinguishable.

Arrangements Around a Circular Table



Let us consider that persons A, B, C, D are sitting around a round table. If all of them (A, B, C, D) are shifted by one place in anticlockwise order, then we will get Fig. (*b*) from Fig.(*a*). Now, if we shift A, B, C, D in anticlockwise order, we will get Fig.(*c*). Again, if we shift them, we will get Fig. (*d*) and in the next time, Fig. (*a*).

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements, thus obtained will be the same, because anticlockwise order of A, B, C, D does not change.

But if A, B, C, D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is n (number of circular arrangements of different things). Hence, the number of circular arrangements of n different things is-

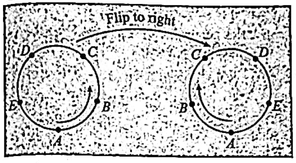
$\dfrac{1}{n}$ (number of linear arrangements of n different things) $= \dfrac{n!}{n} = (n-1)!$

Therefore note that:

The number of circular permutations of n different things taken all at a time is : $(n-1)!$.

Arrangements of beads or flowers (all different) around a circular necklace or garland

Consider five beads A, B, C, D, E in a necklace or five flowers A, B, C, D, E in a garland etc. If the necklace or garland on the left is turned over we obtain the arrangement on the right, i.e., anticlockwise and clockwise order of arrangement is not different we will get arrangements as follows:



We see that arrangements in above figures are not different.

Then the number of circular permutations of n different things taken all at a time is $\dfrac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken as not different.

Number of circular permutations of n different things taken r at a time

CASE-I: If clockwise and anticlockwise orders are taken as different, then the required number of circular permutations $= \dfrac{^nP_r}{r}$.

CASE-II: If clockwise and anticlockwise orders are taken as not different, then the required number of circular permutations $= \dfrac{^nP_r}{2r}$

Train Your Brain

Example 13: Consider 21 different pearls on a necklace. How many ways can the pearls be placed in on this necklace such that 3 specific pearls always remain together?

Sol. After fixing the places of three pearls. Treating 3 specific pearls = 1 units, so we have now 18 pearls + 1 unit = 19 and the number of arrangement will be $(19-1)! = 18!$ also, the number of ways of 3 pearls can be arranged between themselves is $3! = 6$. Since, there is no distinction between the clockwise and anticlockwise arrangements. So, the required number of arrangements $= \dfrac{1}{2}\, 18!\,.\,6 = 3\,(18\,!)$.

Example 14: In how many ways can 24 persons be seated round a table, if there are 13 seats?

Sol. In case of circular table the clockwise and anticlockwise order are different, then the required number of circular permutations $= \frac{{}^{24}P_{13}}{13} = \frac{24!}{13 \times 11!}$.

Example 15: How many necklace of 12 beads each can be made from 18 beads of various colours?

Sol. In the case of necklace there is not distinction between the clockwise and anticlockwise arrangements, then the required number of circular permutations

$$= \frac{{}^{18}P_{12}}{2 \times 12} = \frac{18!}{6! \times 24}$$

$$= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13!}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 24} = \frac{119 \times 13!}{2}.$$

Example 16: In how many ways 10 boys and 5 girls can sit around a circular table so that no two girls sit together.

Sol. 10 boys can be seated in a circle in 9! ways. There are 10 spaces in between the boys, which can be occupied by 5 girls in ${}^{10}P_5$ ways. Hence total numbers of ways are $9!\ {}^{10}P_5$.

Example 17: In how many ways can we arrange 6 different flowers in a circle? In how many ways we can form a garland using these flowers?

Sol. The number of circular arrangements of 6 different flowers $= (6 - 1)! = 120$

When we form a garland, clockwise and anticlockwise arrangements are similar. Therefore, the number of ways of forming garland $= \frac{1}{2}(6-1)! = 60$.

Concept Application

12. 20 persons were invited to a party. In how many ways can they and the host be seated at a circular table? In how many of these ways will two particular persons be seated on either side of the host.

(*a*) 20!, 2! 18!　(*b*) 18!, 2! 20!

(*c*) 19!, 18!　(*d*) 19!, 19!

13. The number of ways in which 4 boys and 4 girls can stand in a circle so that each boy and each girl is one after the other is-

(*a*) 3!. 4!　(*b*) 4!　(*c*) 8!. 4!　(*d*) 7!

14. The number of ways in which 6 red roses and 3 white roses (all roses different) can form a garland so that all the white roses come together is-

(*a*) 2170　(*b*) 2165　(*c*) 2160　(*d*) 2155

Credit:--- Divansh.....

COMBINATIONS

Each of different grouping or selections that can be made by some or all of a number of given things without considering the order in which things are placed in each group, is called combinations.

Counting Formulae for Combinations

The number of combinations (selections or groupings) that can be formed from n different objects taken r at a time is denoted by ${}^nC_r$ and its value is equal to

$${}^nC_r = \frac{n!}{(n-r)!r!} \; (0 \le r \le n)$$

as $${}^nC_r = \frac{{}^nP_r}{r!}$$

As in a permutation the arrangement of r selected objects out of n, is done in r! ways and in combination arrangement in a group is not considered.

In particular

- ${}^nC_0 = {}^nC_n = 1$ i.e. there is only one way to select none or to select all objects out of n distinct objects.
- ${}^nC_1 = n$ There are n ways to select one thing out of n distinct things.
- ${}^nC_r = {}^nC_{n-r}$

  Therefore ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$.
- If n is odd then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
- If n is even then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.

Important Results of Combinations (Selections)

- The number of ways in which r objects can be selected from n distinct objects if a particular object is always included is ${}^{n-1}C_{r-1}$.
- The number of ways in which r objects can be selected from n distinct objects if a particular object is always excluded is ${}^{n-1}C_r$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always included is ${}^{n-m}C_{r-m}$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always excluded is ${}^{n-m}C_r$.

Train Your Brain

Example 18: If ${}^{49}C_{3r-2} = {}^{49}C_{2r+1}$, find '$r$'.

Sol. ${}^nC_r = {}^nC_s$ if either $r = s$ or $r + s = n$.

Thus $3r - 2 = 2r + 1 \Rightarrow r = 3$

or $3r - 2 + 2r + 1 = 49 \;\Rightarrow\; 5r - 1 = 49 \;\Rightarrow\; r = 10$

$\therefore r = 3, 10$

Example 19: Fifteen players are there for selection for a cricket match.

(i) In how many ways the playing 11 can be selected?

(ii) In how many ways the playing 11 can be selected including a particular player?

(iii) In how many ways the playing 11 can be selected excluding two particular players?

Sol.

(i) 11 players are to be selected from 15

Number of ways = ${}^{15}C_{11} = 1365$.

(ii) Since one player is already included, we have to select 10 from the remaining 14

Number of ways = ${}^{14}C_{10} = 1001$.

(iii) Since two players are to be excluded, we have to select 11 from the remaining 13.

Number of ways = ${}^{13}C_{11} = 78$.

Example 20: A committee of 5 members is to be formed from 6 men and 4 women so that

(i) At least two women and atleast one man.

(ii) Two particular opposite sex always included.

(iii) Two particular opposite sex refuse to serve in the same committee

(iv) Majority of women serve in the committee.

(v) M_1, M_2, and W are particular men and woman respectively, M_1 refuses M_2, M_2 refuses W to serve in the same committee.

(vi) (M_1, M_2) (W_1, W_2) are two particular men and, women respectively. Whenever M_1, M_2 serve in the same committee, W_1, W_2 do not and vice-versa.

Sol. Before we attempt the question, let us analyse all possible cases and put them in tabular form

(i)

(a) $W(4)$	$M(6)$	(b) $W(4)$	$M(6)$	(c) $W(4)$	$M(6)$
2	3	3	2	4	1

Hence, total selections $= {}^4C_2 \cdot {}^6C_3 + {}^4C_3 \cdot {}^6C_2 + {}^4C_4 \cdot {}^6C_1$

(ii) After including two particular opposite sex, remaining 3, from 8 can be selected in ${}^8C_3$ ways.

(iii) Total ways $= {}^{10}C_5 - {}^8C_3$

(iv) Required ways $= {}^4C_4 \cdot {}^6C_1 + {}^4C_3 \cdot {}^6C_2$

(v) The following cases to be excluded from the total

(a) Committee includes all three M_1, M_2, W

(b) Committee includes M_1, M_2 but not W

(c) Committee includes M_2, W but not M_1.

Hence, total no. of committees

$= {}^{10}C_5 - {}^7C_2 - {}^7C_3 \times 2 = 161$

[(b) and (c) have same number]

(vi) Required ways $= {}^{10}C_5 - {}^6C_1$ $(M_1, M_2, W_1, W_2$ included)

Example 21: A man has 7 relatives 4 of them ladies and 3 gentlemen, his wife also 7 relatives, 3 of them ladies and 4 gentlemen.

(i) In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that 3 of man relatives and 3 of his wife relatives?

(ii) In how many can they invite a dinner party of 6 members so that 3 of man relatives and 3 of his wifes relative with atleast one lady, one gentleman from each of man and his wife.

Sol.

Man		Wife	
4L	3G	3L	4G
3	0	0	3
1	2	2	1
2	1	1	2
0	3	3	0

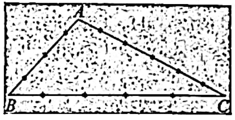
Total $= {}^4C_3 \cdot {}^3C_0 \cdot {}^3C_0 \cdot {}^4C_3 + {}^4C_1 \cdot {}^3C_2 \cdot {}^3C_2 \cdot {}^4C_1 + {}^4C_2 \cdot {}^3C_1 \cdot {}^3C_1 \cdot {}^4C_2 + {}^4C_0 \cdot {}^3C_3 \cdot {}^3C_3 \cdot {}^4C_0 = 485$.

Man		Wife	
4L	3G	3L	4G
1	2	1	2
1	2	2	1
2	1	1	2
2	1	2	1

Total $= {}^4C_1 \cdot {}^3C_2 \cdot {}^3C_1 \cdot {}^4C_2 + {}^4C_1 \cdot {}^3C_2 \cdot {}^3C_2 \cdot {}^4C_1 + {}^4C_2 \cdot {}^3C_1 \cdot {}^3C_1 \cdot {}^4C_2 + {}^4C_2 \cdot {}^3C_1 \cdot {}^3C_2 \cdot {}^4C_1 = 900$

Concept Application

15. The sides AB, BC, CA of ΔABC contains 3, 4, 5 points excluding angular points. By joining these points excluding A, B, C. Find



(i) The number of straight lines

(ii) The number of triangles

(iii) The number of triangles so that no two points on the same side

(iv) The number of quadrilaterals

(v) The number of quadrilaterals whose vertices lie on all sides joining the points.

16. If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$ then the value of r is

(a) 14 (b) 8 (c) 5 (d) 7

17. In how many ways can a committee of 6 be formed out of 6 men and 4 women so that committees include at the most 2 women?

(a) 90 (b) 185 (c) 115 (d) 210

18. A van can accomodate 8 persons, 4 in the front and 4 at the back. If heights of 8 persons are all different, then in how many ways they can be seated in the van so that the person who is sitting in the front is shorter than just at back?

19. An 8 oared boat to be manned from a crew of eleven of whom 3 can only steer but can not row. 8 can row but can not steer. In how many ways the staff can be arranged if 2 of the men can only row on bow side?

TOTAL NUMBER OF SELECTION

Selection From Distinct Objects

The number of ways (or combinations) of selection from n distinct objects, taken at least one of them is

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$$

Logically it can be explained in two ways, as one can be selected in ${}^nC_1$ ways, two in ${}^nC_2$ ways and so on ... and by addition principle of counting the total number of ways of doing either of the job is

$${}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n$$

Also, for every object, there are two choices, either selection or non-selection. Hence total choices are 2^n. But this also includes the case when none of them is selected. Therefore the number of selections, when atleast one is selected $= 2^n - 1$.

Selection From Identical Objects

1. The number of selections of $r(r \leq n)$ objects out of n identical objects is 1.
2. The number of ways of selections of at least one object out of n identical object is n.
3. The number of ways of selections of at least one out of $a_1 + a_2 \ldots + a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, and so on ... a_n are alike of n^{th} kind, is $(a_1 + 1)(a_2 + 1) \ldots (a_n + 1) - 1$.
4. The number of ways of selections of atleast one out of $a_1 + a_2 + a_3 + \ldots + a_n + k$ objects, where a_1 are alike of one kind, ... a_n are alike of n^{th} kind and k are distinct is $(a_1 + 1)(a_2 + 1) \ldots (a_n + 1)2^k - 1$.

Train Your Brain

Example 22: Given five different green dyes, four different blue dyes and three different red dyes, how many combination of dyes can be chosen taking at least one green, one blue dye?

Sol. Any one dye of a particular colour can be either chosen or not; and, thus there are 2 ways in which each one may be dealt with.

Number of ways of selection so that at least one green dye is included = $2^5 - 1 = 31$

(1 is subtracted to correspond to the case when none of the green dyes is chosen.)

A similar argument may be advanced in respect of other two colours also.

Number of combinations = $(2^5 - 1)(2^4 - 1)(2^3)$.

$= 31 \times 15 \times 8 = 3720$

Example 23: Find the number of combinations that can be formed with 5 oranges, 4 mangoes and 3 bananas when it is essential to take

(*i*) At least one fruit

(*ii*) One fruit of each kind.

Sol. Here 5 oranges are alike of one kind, 4 mangoes are alike of second kind and 3 bananas are alike of third kind

(*i*) The required number of combinations (when at least one fruit)

$= (5 + 1)(4 + 1)(3 + 1)2^0 - 1 = 120 - 1 = 119$

(*ii*) The required number of combinations (when one fruit of each kind)

$= {}^5C_1 \times {}^4C_1 \times {}^3C_1 = 5 \times 4 \times 3 = 60.$

Concept Application

20. A basket contains 4 identical apples, 5 identical mangoes, 3 distinct oranges.

(*i*) Find the total number of ways of selecting fruits from the basket.

(*ii*) Find the total number of ways of selecting fruits from the basket

- With at least one fruit
- At least one from each kind

21. There are 12 fruits in a basket of which 5 are apples, 4 mangoes and 3 bananas (fruits of same species: identical).

How many ways are there to select atleast one fruit?

22. In how many ways we can select 4 letters from the letters of the word MISSISSIPPI?

23. The number of words which can be formed using 4 letters of the word 'EXAMINATION' is-

(*a*) 1896 (*b*) 2136 (*c*) 2454 (*d*) 1986

DIVISORS OF A GIVEN NATURAL NUMBER

Let $n \in \mathrm{N}$ and $n = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}...p_k^{\alpha_k}$, where $p_1, p_2, p_3, ... p_k$ are different prime numbers and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k$ are natural numbers then:

- The total number of divisors of N including 1 and n is $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) ... (\alpha_k + 1)$
- The total number of divisors of n excluding 1 and n is $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) ... (\alpha_k + 1) - 2$ (Proper Diviso
- The sum of these divisors is $= (p_1^0 + p_1^1 + p_1^2 + ... + p_1^{\alpha_1})(p_2^0 + p_2^1 + p_2^2 + ... + p_2^{\alpha_2})$ $(p_k^0 + p_k^1 + p_k^2 + ... + p_k^{\alpha_k})$ (Use sum of G.P. in each bracket)
- The number of ways in which n can be resolved as a prod of two factors is $= \frac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1) ... (\alpha_k + 1)$, if n is not a perfect squa or $= \frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1) ... (\alpha_k + 1) + 1]$, if n is a perfect squa
- The number of ways in which composite number n can b resolved into two factors which are relatively prime (o coprime) to each other is equal to 2^{k-1} where k is the numb of different factors (or different primes) in n.

Train Your Brain

Example 24: If $n = 10800$, then find the

(*i*) Total number of divisors of n

(*ii*) The number of even divisors

(*iii*) The number of divisors of the form $4m + 2$

(*iv*) The number of divisors which are multiples of 15

Sol. $n = 10800 = 2^4 \times 3^3 \times 5^2$

Any divisor of n will be of the form $2^a \times 3^b \times 5^c$ where $0 \le a \le 4, 0 \le b \le 3, 0 \le c \le 2.$

For any distinct choices of a, b and c, we get a divisor of n

(*i*) Total number of divisors = $(4 + 1)(3 + 1)(2 + 1) = 60$

(*ii*) For a divisor to be even, a should be at least one. So total number of even divisors = $4(3 + 1)(2 + 1) = 48.$

(*iii*) $4m + 2 = 2(2m + 1)$. In any divisor of the form $4m + 2$, 'a' should be exactly 1. So number of divisors of the form $4m + 2 = 1\ (3 + 1)(2 + 1) = 12.$

(*iv*) A divisor of n will be a multiple of 15 if b is at least one and c is at least one. So number of such divisors $= (4 + 1) \times 3 \times 2 = 30.$

Example 25: Find the number of divisors of 1350. Also find the sum of all divisors.

Sol. $1350 = 2 \times 3^3 \times 5^2$

$\therefore$ Number of divisors = $(1 + 1)(3 + 1)(2 + 1) = 24$

Sum of divisors = $(1 + 2)(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2)$ $= 3720.$

Example 26: In how many ways 8100 can be resolved into product of two factors?

Sol. $8100 = 2^2 \times 3^4 \times 5^2$

Number of ways = $\frac{1}{2}[(2 + 1)(4 + 1)(2 + 1) + 1] = 23$

Example 27: How many functions can be defined from a set A containing 5 elements to a set B having 3 elements? How many of these are surjective functions?

Sol. Image of each element of A can be taken in 3 ways.

$\therefore$ Number of functions from A to $B = 3^5 = 243$.

Number of into functions from A to B

$= 2^5 + 2^5 + 2^5 - 3 = 93$.

$\therefore$ Number of onto functions = 150.

Concept Application

24. Consider the number $n = 2^2 \cdot 3^3 \cdot 5^5 \cdot 7^4 \cdot 11^3$
    (*i*) The total number of divisors of n is
    (*ii*) The number of proper divisors of n is
    (*iii*) The number of odd divisors of n is
    (*iv*) The number of even divisors of n is
    (*v*) The number of ordered pairs (a, b) so that LCM $(a, b) = n$ is
    (*vi*) The sum of divisors of n is
    (*vii*) The number of unordered pairs (x, y) so that $n = xy$ is
    (*viii*) The number of unordered pairs (x, y) so that $n = xy$ and $gcd(x, y) = 1$ is
    (*ix*) The number of divisors of n which are not multiple of 15 is
    (*x*) The number of divisors of n in the form $4k + 2$, $k \geq 0, k \in z$ is
    (*xi*) Then number of divisors of the form $6k + 1$ is

Exponent of Prime p in n!

Exponent of prime p in n! is denoted by $E_p(n!)$, where p is a prime number and n is a natural number. Then the last integer amongst $1, 2, 3, ..., (n - 1), n$ which is divisible by p is $[n/p]p$, where $[n/p]$ denotes the greatest integer. ($\because [x] \leq x$)

$$E_p(n!) = E_p(1, 2, 3 \dots (n-1).n) = E_p\,(p.2p.3p. \dots (n-1)p.[n/p]\,p)$$

Because the remaining natural numbers from 1 to n are not divisible by p.

$$\text{Thus, } \left[\frac{n}{p}\right] + E_p\left(1.2.3\dots\left[\frac{n}{p}\right]\right)$$

Now the last integer among $1, 2, 3, ..., \left[\frac{n}{p}\right]$ which is divisible by p is

$$\left[\frac{n/p}{p}\right] = \left[\frac{n}{p^2}\right] = \left[\frac{n}{p}\right] + E_p\left(p.2p.3p\dots\left[\frac{n}{p^2}\right]p\right)$$

Because the remaining natural numbers from 1 to $\left[\frac{n}{p}\right]$ are not divisible by p.

$$\text{Thus, } \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1.2.3\dots\left[\frac{n}{p^2}\right]\right)$$

Similarly, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots + \left[\frac{n}{p^s}\right]$$

Where s is the largest natural number such that $p^s \leq n < p^{s+1}$

Note: $E_r(n!) \leq E_p(n!)$ (If $r > p$)

Where r and p are prime numbers.

Train Your Brain

Example 28: Find the number of zeros at the end of 100!.

Sol. In terms of prime factors 100! can be written as $2^a.3^b.5^c.7^d \dots$

$$\text{Now} \quad E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$= 50 + 25 + 12 + 6 + 3 + 1 = 97$

$$\text{and } E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$$

$\therefore 100! = 2^{97}.3^b.5^{24}.7^d \dots$

$= 2^{73}.3^b.(2 \times 5)^{24}.7^d \dots = 2^{73}.3^b.(10)^{24}.7^d \dots$

Hence, number of zeros at the end of 100! is 24.

Exponent of 10 in 100! = min (97, 24) = 24.

Example 29: Find the highest power of 3 in 98!.

$$\text{Sol. } E_3(98!) = \left[\frac{98}{3}\right] + \left[\frac{98}{9}\right] + \left[\frac{98}{27}\right] + \left[\frac{98}{81}\right]$$

$= 32 + 10 + 3 + 1 = 45$

Example 30: For how many values of n, $n!$ ends in 15 zeros?

Sol. Since $[x] \leq x$,

$$\therefore E_5(n!) = \left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \dots < \frac{n}{5} + \frac{n}{5^2} + \dots \infty$$

$$\Rightarrow 15 < \frac{\frac{n}{5}}{1 - \frac{1}{5}} \Rightarrow n > 60$$

Let us check from 61 onwards.

$E_5(64!) = 14$, $E_5(65!) = 15$, $E_5(66!) = 15$

$E_5(67!) = 15$, $E_5(68!) = 15$, $E_5(69!) = 15$,

n can be 65, 66, 67, 68, 69.

Concept Application

25. If $(500)! = k(2^a 3^b 5^c)$ (where $a, b, c \in N$, k does not contain any factor of 2, 3 and 5) then which one is/are correct
    (*a*) $a > c$
    (*b*) $c = 124$
    (*c*) Number of zeros at the end of (500)! is equal to 124
    (*d*) (500)! is an even number

Credit:--- Divansh.....

DIVISION OF DISTINCT OBJECT INTO GROUPS

In the case of grouping we have the following. If $m + n + p$ things are divided into 3 groups one containing m, the second n and the third p things; number of groupings is ${}^{(m+n+p)}C_m \,.\, {}^{(n+p)}C_n \,.\, {}^{p}C_p$

$$= \frac{(m+n+p)!}{m!\,n!\,p!}$$ where m, n, p are distinct natural numbers.

If $m = n = p$ (say) then the number of groupings (unmindful of the order of grouping) is $\frac{3m!}{(m!)^3 3!}$

Thus if 52 cards be divided into four groups of 13 each, the number of groupings is $\frac{52!}{(13!)^4 . 4!}$

On the other hand, when 52 cards are dealt 13 each to four persons, the number of ways in which this can be done is $\frac{52!}{(13!)^4}$

As another example, if we consider the division of 52 cards into four groups, three groups containing each 16 and the fourth cards, the number of ways in which this can be done is $\frac{52!}{3!(16!)^3 4!}$

Note: the 3! factor in the denominator. This is for the reason that there are only 3 equal groups.

In general, the number of ways in which mn different things can be divided equally into m distinct groups is $\frac{(mn)!}{(n!)^m}$ when order of groups is important.

Whereas when order of groups is not important, then division into m equal group is done in $\frac{(mn)!}{m!(n!)^m}$ ways.

In general, we can write formula for grouping as factorial of total number of elements divided by product of factorial of number of elements in each group and product of factorial of number of groups having same number of elements, if any. Also formula for number of distribution is number of grouping multiplied with factorial of number of persons in which objects are distributed, divided by factorial of number of persons who got nothing, if any.

Train Your Brain

Example 31: 12 different toys are to be distributed to three children equally. In how many ways this can be done?

Sol. The problem is to divide 12 different things into three different groups.

Number of ways $= \frac{12!}{4!4!4!3!} \times 3! = 34650.$

Example 32: In how many ways can 15 students be divided into 3 groups of 5 students each such that 2 particular students are always together? Also find the number of ways if these groups are to be sent to three different colleges.

Sol. Out of remaining 13 students we can select 3 students in ${}^{13}C_3$ ways.

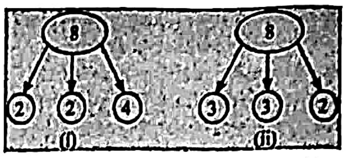
Now total ways of forming groups of 5, 5, 5

$$= \frac{{}^{13}C_3 \times {}^{10}C_5 \times {}^{5}C_5}{2!} = \frac{13!}{3!(5!^2)\cdot 2!}$$

Now if these groups are to be sent to three different colleges, the total number of ways $= \frac{13!}{3! \times 5! \times 5! \times 2!} \times 3!$

Example 33: In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books?

Sol. If each receives at least two books, then the division trees would be as shown below:



The number of ways of division for tree in figure (i) is

$$\left[\frac{8!}{(2!)^2 4!2!}\right].$$

The number of ways of division for tree in figure (ii) is

$$\left[\frac{8!}{(3!)^2 \cdot 2! \cdot 2!}\right].$$

The total number of ways of distribution of these groups among 3 students is $\left[\frac{8!}{(2!)^2 4!2!} + \frac{8!}{(3!)^2 2!2!}\right] \times 3!$

Concept Application

26. The number of ways in which six different prizes can be distributed among three children each receiving at least one prize is
(a) 270 (b) 540 (c) 1080 (d) 2160

27. The number of ways in which mn students can be distributed equally among n sections is
(a) $(mn)^n$ (b) $\frac{(mn)!}{(m!)^n}$ (c) $\frac{(mn)!}{m!}$ (d) $\frac{(mn)!}{m!n!}$

28. Number of ways in which a pack of 52 playing cards be distributed equally amount four players, so that each may have the ace, king, queen and jack of the same suit is-
(a) $\frac{36!}{(9!)^4}$ (b) $\frac{36!4!}{(9!)^4}$
(c) $\frac{36!}{(9!)^4 4!}$ (d) $\frac{36!}{(9!)^4(4!)^4}$

29. Note that number of ways in which 30 Jawans can be divided into 3 equal groups

30. The number of ways in which 30 Jawans can be deputed equally on 3 borders. (10 on each border) is

USE OF MULTIONOMIALS

- If there are l objects of one kind, m objects of second kind, n objects of third kind and so on; then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + ...$) is the coefficient of x^r in the expansion of $(1 + x + x^2 + x^3 + ... x^l)(1 + x + x^2 + ... + x^m)(1 + x + x^2 + ... + x^n)$

  Further if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + ...$) is the coefficient of x^r in the expansion of $(x + x^2 + x^3 + ... x^l)(x + x^2 + x^3 + ... + x^m)(x + x^2 + x^3 + ... + x^n)$

- If there are l objects of one kind, m object of second kind, n object of third kind and so on; then the number of possible arrangements/permutations of r objects out of these objects (i.e., $l + m + n + ...$) is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+........+\frac{x^l}{l!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+.......+\frac{x^m}{m!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+.......+\frac{x^n}{n!}\right)...$$

Note: For use in problems of the above type the following Binomial expansions may be noted

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + ... + x^r + ...$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + ... + (r+1)x^r + ...$$

$$\frac{1}{(1-x)^3} = (1-x)^{-3} = 1 + 3x + 6x^2 + ... + \frac{(r+1)(r+2)}{1.2}x^r + ...$$

$$\frac{1}{(1-x)^4} = (1-x)^{-4} = 1 + 4x + 10x^2 + ... + \frac{(r+1)(r+2)(r+3)}{1.2.3}x^r + ...$$

and more generally $\frac{1}{(1-x)^p} = (1-x)^{-p} = 1 + px + ... + \frac{(r+1)(r+2)...(r+\overline{p-1})}{1.2.3...(p-1)}x^r + ...$

and the coefficient of x^r in this general case is easily seen to be ${}^{(r+p-1)}C_{p-1}$

An Alternative Method for the General Problem

In this case, we consider additionally $(r - 1)$ things (this number is one less than r, the number of persons). It may be seen that the number of ways of dividing the n things among the r persons as per the condition of the problem is ${}^{(n+r-1)}C_{(r-1)}$.

Use of Multinomials in Solving Linear Equation

Multinomials can be used to solve linear equations. The method is illustrated with the help of following illustrations.

e.g. **Find the number of non negative integral solutions of $x + y + z + w = 20$.**

Any one of the four variables can take from the value zero to the value 20; and hence we construct a polynomial in a variable, say x, with x raised to different powers which would constitute the values that any one variable can take when the equation is solved in the manner indicated. We, thus, consider the product expression.

$(1 + x + x^2 + ... + x^{20})(1 + x + ... + x^{20})(1 + x + ... + x^{20})(1 + x + ... + x^{20})$

Four factors since there are four variables. If we take x^4 in the first factor, x^5 in the second; x^8 in the third, then we take the term x^3 in the fourth so that the sum of the powers $(4 + 5 + 8 + 3 = 20)$ is 20. It is then we say that there is a solution corresponding to $x = 4$, $y = 5$; $z = 8$; $w = 3$. Hence the number of solutions in the manner required is the coefficient of x^{20} in

$(1 + x + ... + x^{20})^4$

i.e. in $\left(\frac{1-x^{21}}{1-x}\right)^4$

i.e. in $(1 - x^{21})^4(1 - x)^{-4}$

i.e. in $(1 - x)^{-4}$ and this is $= {}^{23}C_3$

(Note: In $(1 - x)^{-4}$ coefficient of x^r is ${}^{(r+3)}C_3$

Method of Fictious Partition

Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is ${}^{n+p-1}C_n$.

Train Your Brain

Example 34: Let us consider the more general problem of distributing n identical things given among r persons, each one whom, can receive 0, 1, 2 or more things $(\leq n)$.

Sol. Consider r brackets corresponding to the r persons.

In each bracket take an expression given by $1 + x + x^2 + ... + x^n$ (the various powers of x, namely 0, 1, 2, ..., n correspond to the number of things each person can have in the distribution)

In the continued product $(1 + x + x^2 + ... + x^n)(\ )(\ )$ ... repeated r times, collect the coefficient of x^n. This coefficient gives the required number of ways of distribution.

∴ The number of ways

= Coefficient of x^n in $(1 + x + x^2 + ... + x^n)(\ )(\ )$ ... repeated r times

= Coefficient of x^n in $\left(\frac{1-x^{n+1}}{1-x}\right)^r$

= Coefficient of x^n in $(1 - x^{n+1})^r(1 - x)^{-r}$

$= \text{Coefficient of } x^n \text{ in } (1-x)^{-r}$

$$= \frac{(r+1)(r+2)\ldots(r+n-1)}{1.2\ldots(r-1)} = {}^{(n+r-1)}C_{r-1}$$

Example 35: An unlimited number of Red, White, Blue and Green balls are given. Ten balls are drawn. Find the number of ways of selection.

Sol. In this the expression in x to be considered is $1 + x + x^2 + \ldots$ to correspond to the unlimited number available in each colour. There are four such colours.

The required number of ways

$= \text{Coefficient of } x^{10} \text{ in } [1 + x + x^2 + \ldots]^4$

$$= \text{Coefficient of } x^{10} \text{ in } \left(\frac{1}{1-x}\right)^4 = \frac{11 \times 12 \times 13}{6} = 286$$

Example 36: Find the number of ways in which 11 identical apples can be distributed among 6 children, so that each child receives at least one apple.

Sol. First give one apple to each child. There are remaining 5 apples to be distributed among 6 children (so that each may receive any number of apples not exceeding five).

The number of ways required

$= {}^{(11-1)}C_{6-1} = {}^{10}C_5 = 252$

Alternatively, the number of ways of distribution is given by the coefficient of x^{11} in $(x + x^2 + \ldots + x^{11})^6$.

Example 37: Find the number of solutions of the equation $x + y + z = 6$, where $x, y, z \in W$.

Sol. Number of solutions = coefficient of x^6 in $(1 + x + x^2 + \ldots x^6)^3$

$= \text{coefficient of } x^6 \text{ in } (1-x^7)^3 (1-x)^{-3}$

$= \text{coefficient of } x^6 \text{ in } (1-x)^{-3} = {}^{3+6-1}C_2 = {}^8C_2 = 28.$

Example 38: Let n and k be positive integers such that $n \geq \frac{k(k+1)}{2}$. Find the number of solutions $(x_1, x_2, \ldots x_k)$, $x_1 \geq 1, x_2 \geq 2, \ldots, x_k \geq k$, all integers satisfying

$x_1 + x_2 + \ldots + x_k = n$

Sol. The number of solutions is the coefficient of x^n in $(x + x^2 + \ldots + x^n)(x^2 + x^3 + \ldots + x^n) \ldots (x^k + x^{k+1} \ldots + x^n)$

i.e. coefficient of x^n in

$$x^{\frac{k(k+1)}{2}} \frac{(1-x^n)}{1-x} \frac{(1-x^{n-1})}{1-x} \ldots \frac{(1-x^{n-k+1})}{1-x}$$

$$\text{or } x^{\frac{k(k+1)}{2}} (1-x^n)(1-x^{n-1}) \ldots (1-x^{n-k+1})(1-x)^{-k}$$

$$= \text{coefficient of } x^{n-\frac{k(k+1)}{2}} \text{ in } (1-x)^{-k}$$

$$= {}^{\left(n-\frac{k(k+1)}{2}+k-1\right)}C_{k-1}$$

(Note: Coefficient of x^n in $(1-x)^{-k}$ is ${}^{(n+k-1)}C_{k-1}$)

Credit:--- Divansh.....

Concept Application

31. For the equation $x + y + z + w = 19$, the number of positive integral solutions is equal to-
    (a) The number of ways in which 15 identical things among 4 persons can be distributed
    (b) The number of ways in which 19 identical things can be distributed among 4 persons
    (c) The number of ways in which 16 identical things among 3 persons can be distributed
    (d) The number of ways in which 14 identical things among 4 persons can be distributed
32. Three distinguishable dice are rolled. In how many ways we can get a total 15?
33. In how many ways we can give 5 apples, 4 mangoes and 3 oranges (fruits of same species are similar) to three persons if each may receive none, one or more?
34. The number of non-negative integral solutions of $x_1 + x_2 + x_3 + x_4 \le n$ (where n is a positive integer) is-
    (a) ${}^{n+3}C_3$ (b) ${}^{n+4}C_4$
    (c) ${}^{n+5}C_5$ (d) ${}^{n+4}C_n$
35. Find the number of integral solutions of $x + y + z = 20$, if $x \ge -4, y \ge 1, z \ge 2$.

METHOD OF INCLUSION EXCLUSION

If $A_1, A_2, ..., A_m$ are finite sets and $A = A_1 \cup A_2 \cup ... \cup A_m$, then

$n(A) = a_1 - a_2 + a_3 - a_4 + ... + (-1)^{m+1} a_m$

where $a_1 = n(A_1) + n(A_2) + ... + n(A_m)$

$$a_2 = \sum_{1 \le i < j < m} n(A_i \cap A_j)$$

$$a_3 = \sum_{1 \le i < j < k \le m} n(A_i \cap A_j \cap A_k)$$

and so on.

Corollary (Sieve-Formula)

If $A_1, A_2, ... A_m$ are m subsets of a set A containing N elements, then

$$n(A'_1 \cap A'_2 \cap ... \cap A'_m) = N - \sum_i n(A_i) + \sum_{1 \le i < j \le m} n(A_i \cap A_j)$$

$$- \sum_{1 \le i < j < k \le m} n(A_i \cap A_j \cap A_k) + ... + (-1)^m n\ (A_1 \cap A_2 \cap ..... \cap A_m)$$

DEARRANGEMENTS

It is rearrangement of objects such that no one goes to its original place.

If n things are arranged in a row, the number of ways in which they can be rearranged so that none of them occupies its original placed is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} .... + (-1)^n \frac{1}{n!}\right)$$

It is denoted by $D_n = n! \sum_{r=0}^{n} (-1)^r \frac{1}{r!}$

SUMMATION OF NUMBERS (3 DIFFERENT WAYS)

(i) Sum of all the numbers greater then 10000 formed by the digits 1, 3, 5, 7, 9 if no digit being repeated.

Method 1: All possible numbers = 5! = 120

If one occupies the units place then total number = 24.

Hence 1 enjoys units place 24 times

				1

Similarly 1 enjoys each place 24 times

Sum due to $1 = 1 \times 24\,(1 + 10 + 10^2 + 10^3 + 10^4)$

Similarly

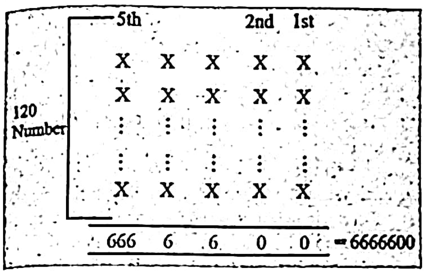
Sum due to the digit $3 = 3 \times 24(1 + 10 + 10^2 + 10^3 + 10^4)$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

Required total sum $= 24\,(1 + 10 + 10^2 + 10^3 + 10^4)$
$(1 + 3 + 5 + 7 + 9)$

Method 2: In 1st column there are twenty four 1's, Twenty four 3's and so on and their sum is = 24 × 25 = 600

Hence add. in vertical column normally we get = 6666600



Method 3: Applicable only if the digits used are such that they have the same common difference. (valid even if the digits are repeating) Writing all the numbers in ascending order of magnitude

$S = (13579 + 13597 + ... + 97513 + 97531)$

$S = (13579 + 99531) + (13597 + 97513) + ...$

$= (111110)$ 60 time = 6666600 Ans.

$S = \frac{n}{2}(l + L)$ where n = number of numbers, l = smallest, L = Largest

Train Your Brain

Example 39: A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) at least two of them are in the wrong envelopes. (ii) all the letters are in the wrong envelopes.

Sol. (i) The number of ways in which at least two of them are in the wrong envelopes

$$= \sum_{r=2}^{6} {}^nC_{n-r} D_r$$

$$= {}^nC_{n-2}D_2 + {}^nC_{n-3}D_3 + {}^nC_{n-4}D_4 + {}^nC_{n-5}D_5 + {}^nC_{n-6}D_6$$

Here $n = 6$,

$$= {}^6C_4 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$$

$$+ {}^6C_2 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$+ {}^6C_1 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

$$+ {}^6C_0 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$= 15 + 40 + 135 + 264 + 265 = 719.$

(ii) The number of ways in which all letters be placed in wrong envelopes

$$= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$= 720\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$$

$= 360 - 120 + 30 - 6 + 1 = 265.$

Example 40: Four slip of papers with the numbers 1, 2, 3, 4 written on them are put in a box. They are drawn one by one (without replacement) at random. In how many ways it can happen that the ordinal number of atleast one slip coincide with its own number?

Sol. Total number of ways = 4 ! = 24.

The number of ways in which ordinal number of any slip does not coincide with its own number is the number of dearrangements of 4 objects $= 4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$

Thus the required number of ways = 24 − 9 = 15

Concept Application

36. The number of ways to put five letters in five envelopes when any one letter is kept in right envelope and four letters in wrong envelopes are
    (a) 40 (b) 45 (c) 30 (d) 70
37. There are four balls of different colours and four boxes of colours same as those of the balls. The number of ways in which the balls, one in each box, could be placed such that a ball does not go to box of its own colour is
    (a) 8 (b) 7 (c) 9 (d) 6
38. The sum of all numbers greater than 1000 formed by using the digits 1, 3, 5, 7 no digit is repeated in any number is
    (a) 106656 (b) 101276
    (c) 117312 (d) 311273
39. The sum of the digits in the unit place of all the numbers formed with the help of 3, 4, 5, 6 taken all at a time is
    (a) 18 (b) 108 (c) 432 (d) 180

Credit:--- Divansh.....

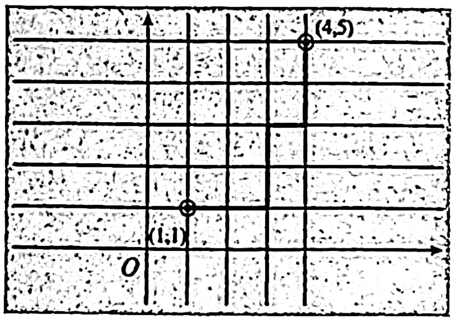
GRID PROBLEM

Train Your Brain

Example 41: Number of ways in which an ant can reach from (1, 1) to (4, 5) via shortest path.

Sol. Whatever may be the mode of travel of the ant; it has to traverse 3H (Horizontal) and 4V (Vertical) paths.

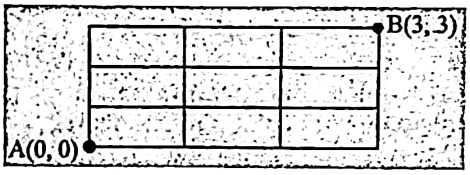
Hence required number of ways $= \dfrac{7!}{4!3!} = {}^7C_3$



Note: If there are n vertical and m horizontal lines then there will be $(n-1)$ horizontal and $(m-1)$ vertical paths.

Concept Application

40. In how many ways can a person go from point A to point B if he can travel only to the right or upward along the lines (Grid Problem)?



41. The streets of a city are arranged like the lines of a chess board. There are m streets running North to South and 'n' streets running East to West. The number of ways in which a man can travel from NW to SE corner going the shortest possible distance is-

(*a*) $\sqrt{m^2+n^2}$

(*b*) $\sqrt{(m-n)^2\cdot(n-1)^2}$

(*c*) $\dfrac{(m+1)!}{m!\cdot n!}$

(*d*) $\dfrac{(m+n-2)!}{(m-1)!\cdot(n-1)!}$

Some Important Results About Points

If there are n points in a plane which m ($< n$) are collinear, then

(*a*) The number of different straight lines obtained by joining these n points is
${}^nC_2 - {}^mC_2 + 1$

(*b*) Total number of different triangles formed by joining these n points is
${}^nC_3 - {}^mC_3$

(*c*) Number of diagonals in polygon of n sides is .
${}^nC_2 - n$ i.e. $\dfrac{n(n-3)}{2}$

(*d*) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed is
${}^mC_2 \times {}^nC_2$ i.e. $\dfrac{mn\,(m-1)\,(n-1)}{4}$

(*e*) Number of triangles formed by joining vertices of convex polygon of n sides is ${}^nC_3$ of which

- (*i*) Number of triangles having exactly two sides common to the polygon $= n$
- (*ii*) Number of triangles having exactly one side common to the polygon $= n(n-4)$
- (*iii*) Number of triangles having no side common to the polygon $= \dfrac{n(n-4)\,(n-5)}{6}$

Credit:--- Divansh.....

Aarambh (Solved Examples)

1. There are m points on one straight line AB and n points on another straight line AC, none of them being A. How many triangles can be formed if point A is also included?

(a) $\dfrac{mn(m+n)}{2}$ (b) $\dfrac{mn(m+n)}{3}$

(c) $\dfrac{mn(m+n)}{4}$ (d) $\dfrac{mn(m+n)}{5}$

Sol. To get a triangle, we either take two points on AB and one point on AC, or one point on AB and two points on AC. Therefore, the number of triangles we obtain

$$= ({}^mC_2)({}^nC_1) + ({}^mC_1)({}^nC_2)$$

$$= \frac{m(m-1)}{2}n + m\frac{n(n-1)}{2}$$

$$= \frac{1}{2}mn(m-1+n-1) = \frac{1}{2}mn(m+n-2)$$

If the point A is included, additional triangles, we get mn additional triangles.

Thus, in this case we get $= \dfrac{mn}{2}(m+n-2)+mn$

$= \dfrac{mn(m+n)}{2}$ triangles

Therefore, option (a) is the correct answer.

2. In an examination, the maximum marks for each of three papers is n and that for the fourth paper is $2n$. The number of ways in which a candidate can get $3n$ marks is

(a) $\dfrac{1}{5}(n+1)(5n^2+10n+6)$

(b) $\dfrac{1}{6}(n+1)(5n^2+10n+6)$

(c) $\dfrac{1}{3}(n+1)(5n^2+10n+6)$

(d) None of these

Sol. The number of ways of getting $3n$ marks

$= \text{coefficient of } x^{3n} \text{ in } (1+x+x^2+\ldots+x^n)^3 \cdot (1+x+\ldots+x^{2n})$

$= \text{coefficient of } x^{3n} \text{ in } (1-x^{n+1})^3 (1-x^{2n+1}) \cdot (1-x)^{-4}$

$= \text{coefficient of } x^{3n} \text{ in } (1 - 3x^{n+1} + 3x^{2n+2} - x^{3n+3})$
$(1-x^{2n+1}) \times (1 + {}^4C_1 x + {}^5C_2 x^2 + {}^6C_3 x^3 + \ldots)$

$= \text{coefficient of } x^{3n} \text{ in } (1 - 3x^{n+1} - x^{2n+1} + 3x^{2n+2})$
$(1 + {}^4C_1 x + {}^5C_2 x^2 + \ldots)$

$$= {}^{3n+3}C_{3n} - 3 \cdot {}^{2n+2}C_{2n-1} + 3 \cdot {}^{n+1}C_{n-2} - {}^{n+2}C_{n-1}$$

$$= \frac{(3n+3)!}{3!(3n)!} - 3\cdot\frac{(2n+2)!}{3!(2n-1)!} + 3\frac{(n+1)!}{3!(n-2)!} - \frac{(n+2)!}{3!(n-1)!}$$

$$= \frac{1}{6}(n+1)(27n^2+27n+6-24n^2-12n+3n^2-3n-n^2-2n)$$

$$= \frac{1}{6}(n+1)(5n^2+10n+6)$$

Therefore, option (b) is the correct answer.

3. We have 21 identical balls available with us which we need to be distributed amongst 3 boys A, B and C such that A always gets an even number of balls. The number of possible ways of doing this is

(a) 112 (b) 120 (c) 126 (d) 132

Sol. We have to find non negative integral solution of equation $2x+y+z=21$

Note that x can take a maximum value of 10 and a minimum value of 0.

We rearrange equation so that we get an integer equation with y and z as variables, x as a constant

$y+z=21-2x$

The number of non negative integer solution is

${}^{21-2x+2-1}C_1 = {}^{22-2x}C_1 = 22-2x$

We now add the number of solutions so obtained for all the possible values of x.

The total number of solution is therefore

$$\sum_{x=0}^{10}(22-2x) = 132$$

Therefore, option (d) is the correct answer.

4. All the 7 digit numbers containing each of the digits 2, 3, 5, 6, 7, 8, 9 exactly once and not divisible by 5 are arranged in increasing order. The 2017th number in the list is

(a) 6298753 (b) 6325789

(c) 6352789 (d) 6252789

Sol. Number beginning with 2 and not divisible by 5 is

$6! - 120 = 600$

Number beginning with 3 and not divisible by 5 = 600

Number beginning with 5 and not divisible by 5 = 720

Total = 1920

Number beginning with 6 2 and not divisible by 5 is

$120 - 24 = 96$

∴ Total = 2016

∴ 2017th number is 6325789

Therefore, option (b) is the correct answer.

5. Find the number of times of the digit 4 will be written when listing the integers from 1 to 1000.

Sol. Since 4 does not occur in 1000, we have to count the number of times 4 occurs when we list the integers from 1 to 999. Any number between 1 and 999 is of the form xyz where

$0 \le x, y, z, \le 9.$

Let us first count the numbers in which 4 occurs exactly once. i.e. $({}^3C_1)(9)(9) = 243$ and 4 can occurs exactly at two places in $({}^3C_2)(9) = 3 \times 9$ such number.

Lastly, 4 can occur in all three digits in one number only. Hence the number of times 4 occurs is

$3 \times 9 \times 9 + 3 \times 9 \times 2 + 1 \times 3 = 300.$

Therefore, [300] is the correct answer.

6. How many integral solutions are there to the system of equations $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$ when $x_k \ge 0$?

Credit:--- Divansh.....

Sol. We have: $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$

These two equations reduce to $x_4 + x_5 = 15$ ...(i)

and, $x_1 + x_2 + x_3 = 5$ ...(ii)

Since corresponding to each solution of (i) there are solutions of equation (ii). So, total number of solutions of the given system of equations

= No. of solutions of (i) × No. of solutions of (ii)

$= {}^{15+2-1}C_{2-1} \times {}^{5+3-1}C_{3-1} = {}^{16}C_1 \times {}^{7}C_2 = 336.$

Therefore, [336] is the correct answer.

7. How many integers between 1 and 10^6 have the sum of their digits equal to 18?

Sol. Let $n = x_6\, x_5\, x_4\, x_3\, x_2\, x_1$ be a number between 1 and 10, where $x_1, x_2, x_3 \ldots x_6$ are digits at one's, tens, hundreds... places. Clearly $0 \le x_i \le 9, i = 1, 2, \ldots, 9.$

The total number of values of n is the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$, where $0 \le x \le 9; i = 1, 2, \ldots\ldots 9$

The number of solutions of the above equation is

Coefficient of x^{18} in $(x^0 + x^1 + x^2 + \ldots + x^9)^6$

$= \text{Coefficient of } x^{18} \text{ in } \left(\frac{1-x^{10}}{1-x}\right)^6$

$= \text{Coefficient of } x^{18} \text{ in } (1 - x^{10})^6 (1 - x)^{-6}$

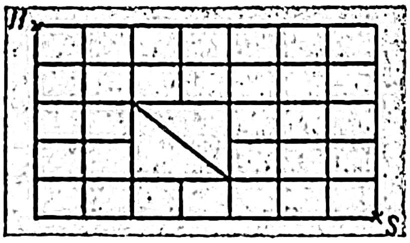
Coefficient of x^{18} in $(1 - {}^6C_1 x^{10} + {}^6C_2 x^{20} \ldots)(1 - x)^{-6}$

= Coefficient of x^{18} in $(1 - x)^{-6} - {}^6C_1$ × Coefficient of x^8 in $(1 - x)^{-6} = {}^{18+6-1}C_{6-1} - C_1 \times {}^{8+6-1}C_{6-1} = {}^{23}C_5 - 6 \times {}^{13}C_5$

$= 33649 - 7722 = 25927$

Therefore, [25927] is the correct answer.

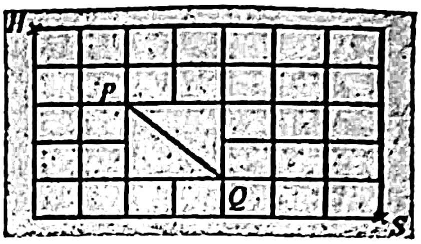
8. The figure shows the plan of a town where all the street blocks are square. In the middle of the town is a park with a diagonal road through it. Albertos walks every day from her house at H to her school at S, always taking one of the shortest routes. Find the number of different shortest routes that she can choose.



Sol. Number of way from H to $P = \dfrac{4!}{2!2!} = 6$

Number of ways from P to Q = 1 (obvious)

Number of ways from Q to $S = \dfrac{4!}{1!3!} = 4$



Total ways = 6 × 4 = 24.

Therefore, [24] is the correct answer.

9. A palindrome is a word or number that reads the backwards as forwards; e.g., LEVEL or 9889 or 121. If sum of all the three digit palindromes are 100 K, then the value of K.

Sol. Total 3 digit palindromes are 9 × 10 × 1

We have to find sum of these 90 numbers

Now sum due to unit place

$1 \times 10(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 45 \times 10$

Sum due to 10th place

$9 \times 10(1 + 2 + 3 + 4 + 4 + 5 + 6 + 7 + 8 + 9) = 45 \times 10 \times$

Sum due to 100th place

$10 \times 10^2(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 45 \times 10 \times 10$

$\therefore S = SU + ST + SH$

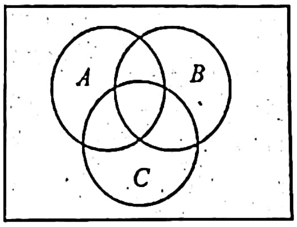
$(1) + (2) + (3) = 45 \times 10(1 + 9 + 10^2)$

$= 45 \times 10 \times 110 = 495 \times 10^2$

Therefore, [495] is the correct answer.

10. The total number of 7 digit numbers that can be formed using two 3's, two 4's, two 5's and one 6 so that no two consecutive digits are identicals is:

Sol. In such of type of examples we can proceed through Venn diagram: Let A denotes region in which two 3's are together. Let B denotes region in which two 4's are together. Let C denotes region in which two 5's are together



$n(A \cap B \cap C)$ = Total number of ways in which 33, 44 and 55 are together = 4! = 24

$n(A \cap B) = \dfrac{5!}{2} = 60$, Also $n(A) = \dfrac{6!}{2!2!} = 180$

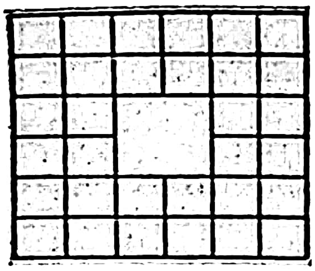
∴ Total number of ways in which no two consecutive digits are identical = Total – Invalid

$= \dfrac{7!}{2!2!2!} - (3 \times 180 - 3 \times 60 + 24) = 630 - 384 = 246$

Therefore, [246] is the correct answer.

11. The number of rectangles whose edges lie completely on the grid lines of following figure is:

Credit:--- Divansh.....



sol. Total number of rectangles = ${}^7C_2 \times {}^7C_2 - 4(27 + 9)$
$= 441 - 144 = 297$
(Using principle of inclusion and exclusion)

Therefore, [297] is the correct answer.

12. Let A_i (where i = 1, 2, 3, ... 12) are the vertices of regular dodecagon and G is its centre. Let S denotes number of straight lines that can be formed with these 13 points and T denotes the number of triangles formed with these points and D denotes the number of diagonals in the dodecagon. then $\left[\frac{T+D+S}{T-D-S}\right]$ is (where [·] denotes greatest integer).

Sol. $S = {}^{12}C_2 = 66$; $T = {}^{13}C_3 - 6 = 286 - 6 = 280$
and $D = {}^{12}C_2 - 12 = 54$
$\therefore T + D + S = 400$ and $T - D - S = 160$
$\therefore \left[\frac{T+D+S}{T-D-S}\right] = 2$

Therefore, [2] is the correct answer.

13. Let N be the number of ordered 6-tuples $(x_1, x_2, x_3, x_4, x_5, x_6)$ of positive integers satisfying $x_1 + x_2 + x_3 + 3x_4 + 3x_5 + 5x_6 = 21$, then $\sqrt{N}$ is equal to

Sol. $x_1 + x_2 + x_3 = x$, $x_4 + x_5 = y$, $z = x_6$
$x + 3y = 21 - 5x_6$ $x_6 = 1, 2$
$x + 3y = 16, 11$
$(x, y, x_6) = (4, 4, 1), (7, 3, 1), (10, 2, 1), (5, 2, 2)$
Number of 6-tuples
$= {}^{4-1}C_2\,{}^{4-1}C_1 + {}^{7-1}C_2\,{}^{3-1}C_1 + {}^{10-1}C_2\,{}^{2-1}C_1 + {}^{5-1}C_2\,{}^{2-1}C_1 = 9 + 30 + 36 + 6 = 81$
$\therefore \sqrt{N} = 9$

Therefore, [9] is the correct answer.

14. Let N is the number of 4 element subsets $\{a, b, c, d\}$ of $\{1, 2, 3, 4, \ldots 20\}$ such that $a + b + c + d$ is divisible by 3. Then the sum of the digits of N is equal to

Sol. Number of ways $= {}^6C_4 + {}^6C_2\,{}^7C_1\,{}^7C_1 + {}^6C_1({}^7C_3 + {}^7C_3) + {}^7C_2\,{}^7C_2$
either all of form $3k$
or two of the form $3k$, one of $3k + 1$, one of $3k + 2$
or one of form $3k$ and 3 of $3k + 1$
or one of form $3k$ and 3 of $3k + 2$
or two of form $3k$ and two of $3k + 2$
$= 15 + 735 + 420 + 441 = 1611$
$\therefore$ sum of digits of $N = 9$

Therefore, [9] is the correct answer.

> *Optimism is turning every 'Oops!' into a 'Wow, that's a learning experience!*

School Level Problems

SINGLE CORRECT TYPE QUESTIONS

1. 10 person are to be arranged around a round table. 3 persons wish to sit as a group number of ways the arrangement can be made is________

(*a*) 9! × 3! (*b*) 8! × 3!

(*c*) 7! × $8P_3$ (*d*) 7! × 3!

2. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is ________

(*a*) {1, 2, 3} (*b*) {1, 2, 3, 4}

(*c*) {1, 2, 3, 4, 5} (*d*) {1, 2, 3, 4, 5}

3. 4 boys picked up 30 mangoes. In how many ways can they divide them, if all mangoes be identical?

(*a*) ${}^{33}C_4$ (*b*) ${}^{33}C_2$

(*c*) 5456 (*d*) 6554

4. The number of ways in which a committee of 3 women and 4 men be chosen from 8 women and 7 men is formed if Mrs. *A* refuses to serve on the committee if Mr. *B* is a member of the committee is

(*a*) 420 (*b*) 840

(*c*) 1540 (*d*) None of these

5. The number of solutions of the inequality ${}^{10}C_{x-1} > 3.\ {}^{10}C_x$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 9

6. A library has '*a*' copies of one book, '*b*' copies of two books, '*c*' copies of each of three books and single copy of '*d*' books. The total no. of ways in which these books can be distributed is________

(*a*) $\dfrac{(a+2b+3c+d)!}{a!b!c!}$ (*b*) $\dfrac{(a+2b+3c+d)!}{a!b^2!c^3!}$

(*c*) $\dfrac{(a+2b+3c+d)!}{a!(b!)^2(c!)^3}$ (*d*) $\dfrac{a+2b+3c+d}{a+b+c+d}$

7. If $\dfrac{1}{{}^4C_n} = \dfrac{1}{{}^5C_n} + \dfrac{1}{{}^6C_n}$, then value of *n* is ________

(*a*) 3 (*b*) 4

(*c*) 0 (*d*) None of these

8. The number of integer *a*, *b*, *c*, *d*, such that $a + b + c + d = 24$ and $a, b, c, d \geq 0$ is

(*a*) ${}^{24}C_3$ (*b*) ${}^{25}C_3$

(*c*) ${}^{26}C_3$ (*d*) ${}^{27}C_3$

9. The number of ways in which 10 persons can go in two cars so that there may be 5 in each car supposing that two particular persons will not go in the same car is:

(*a*) $\dfrac{1}{2}({}^{10}C_5)$ (*b*) $\dfrac{1}{2}({}^{8}C_5)$

(*c*) $2({}^{8}C_4)$ (*d*) ${}^{8}C_4$

VERY SHORT ANSWER TYPE QUESTIONS

10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?

11. If $\dfrac{1}{8!} + \dfrac{1}{9!} = \dfrac{x}{10!}$, find *x*.

12. If ${}^nC_9 = {}^nC_8$, find ${}^nC_{17}$.

13. How many words can be formed each of 2 vowels and 3 consonants from the letters of the given word "DAUGHTER"?

14. How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed?

15. Ten buses are plying between two places *A* and *B*. In how many ways a person can travel from *A* to *B* and come back?

16. In a certain city, all telephone numbers have six digits, the first two digits always being 41 or 42 or 46 or 62 or 64. How many telephone numbers have all six digits distinct?

LONG ANSWER TYPE QUESTIONS

17. A sports team of 11 students is to be constituted, choosing at least 5 from Class XI and atleast 5 from Class XII. If there are 20 students in each of these classes, in how many ways can the team be constituted?

18. Find *r*, if $5\,{}^4P_r = 6\,{}^5P_{r-1}$.

19. Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements.

(*i*) Do the words start with *P*

(*ii*) Do all the vowels always occur together

20. A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has

(*i*) no girls

(*ii*) at least one boy and one girl

(*iii*) at least three girls.

21. What is the number or ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(*i*) Four cards are of the same suit

(*ii*) Four cards belong to four different suits

Credit:--- Divansh.....

CASE STUDY BASED QUESTIONS

Case Study-I

Republic day is a national holiday of India. It honours the date on which the constitution of India came into effect on 26 January 1950 replacing the Government of India Act (1935) as the governing document of India and thus, turning the nation into a newly formed republic.

Answer the following question, which are based on the word "REPUBLIC".

22. Find the number of arrangements of the letters of the word 'REPUBLIC'.

(*a*) 40300

(*b*) 30420

(*c*) 40320

(*d*) 40400

23. If the number of arrangements of the letters of the word 'REPUBLIC' is *abcde*, the $(a+b+c+d+e)$ is

(*a*) 10 (*b*) 9

(*c*) 8 (*d*) 15

Case Study-II

Five students Ajay, Shyam, Yojana, Rahul and Akansha are sitting in a playground in a line.

Based on the above information, answer the following questions.

24. Total number of ways of sitting arrangement of five students is

(*a*) 120

(*b*) 60

(*c*) 24

(*d*) None of these

25. Total number of arrangement sitting Yojana and Rahul not sit together, is

(*a*) 72 (*b*) 120

(*c*) 60 (*d*) 144

Prarambh (Topicwise)

FUNDAMENTAL PRINCIPLE OF COUNTING

1. The number of 10-digit numbers such that the product of any two consecutive digits in the number is a prime number, is

(*a*) 1024 (*b*) 2048

(*c*) 512 (*d*) 64

2. The number of six digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6 & 7 so that digits do not repeat and the terminal digits are even is:

(*a*) 144 (*b*) 72

(*c*) 288 (*d*) 720

3. Number of positive integers which have no two digits having the same value with sum of their digits being 45, is

(*a*) 10 ! (*b*) 9 !

(*c*) $9 \cdot 9!$ (*d*) $17 \cdot 8!$

4. The number of signals that can be made with 3 flags each of different colour by hoisting 1 or 2 or 3 above the other, is:

(*a*) 3 (*b*) 7

(*c*) 15 (*d*) 16

LINEAR PERMUTATION AND PERMUTATION UNDER VARIOUS CONDITIONS

5. If all the letters of the word "QUEUE" are arranged in all possible manner as they are in a dictionary, then the rank of the word QUEUE is:

(*a*) 15th (*b*) 16th

(*c*) 17th (*d*) 18th

6. 8 chairs are numbered from 1 to 8. Two women & 3 men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4, then the men select the chairs from among the remaining. The number of possible arrangements is:

(*a*) ${}^6C_3 \cdot {}^4C_4$ (*b*) ${}^6P_2 \cdot {}^4P_3$

(*c*) ${}^4C_3 \cdot {}^4P_3$ (*d*) ${}^4P_2 \cdot {}^6P_3$

7. Number of words that can be made with the letters of the word "GENIUS" if each word neither begins with G nor ends in S, is:

(*a*) 24 (*b*) 240

(*c*) 480 (*d*) 504

Credit:--- Divansh.....

8. 5 boys & 3 girls are sitting in a row of 8 seats. Number of ways in which they can be seated so that not all the girls sit side by side, is:

(*a*) 36000 (*b*) 9080 (*c*) 3960 (*d*) 11600

9. 10 different letters of an alphabet are given. Words with 5 letters are formed from these given letters, then the number of words which have atleast one letter repeated is:

(*a*) 69760 (*b*) 30240 (*c*) 99748 (*d*) 69670

10. In a conference 10 speakers are present. If S_1 wants to speak before S_2 & S_2 wants to speak after S_3, then the number of ways all the 10 speakers can give their speeches with the above restriction if the remaining seven speakers have no objection to speak at any number is:

(*a*) ${}^{10}C_3$ (*b*) ${}^{10}P_8$ (*c*) ${}^{10}P_3$ (*d*) $\frac{10!}{3}$

11. 20 Passengers are to travel by a double decked bus which can accommodate 13 in the upper deck and 7 in the lower deck. The number of ways that they can be divided if 5 refuse to sit in the upper deck and 8 refuse to sit in the lower deck, is

(*a*) 25 (*b*) 21 (*c*) 18 (*d*) 15

12. How many nine digit numbers can be formed using the digits 2, 2, 3, 3, 5, 5, 8, 8, 8 so that the odd digits occupy even positions?

(*a*) 7560 (*b*) 180 (*c*) 16 (*d*) 60

13. The number of words that an be formed by using the letters of the word 'MATHEMATICS' that start as well as end with *T*, is

(*a*) 80720 (*b*) 90720 (*c*) 20860 (*d*) 37528

14. The number of permutations that can be formed by arranging all the letters of the word 'NINETEEN' in which no two E's occur together is.

(*a*) $\frac{8!}{3!3!}$ (*b*) $\frac{5!}{3!\times {}^6C_2}$

(*c*) $\frac{5!}{3!}\times {}^6C_3$ (*d*) $\frac{8!}{5!}\times {}^6C_3$

CIRCULAR PERMUTATION

15. The number of ways in which 6 red roses and 3 white roses (all roses different) can form a garland so that all the white roses come together, is

(*a*) 2170 (*b*) 2165 (*c*) 2160 (*d*) 2155

16. The number of ways in which 5 beads, chosen from 8 different beads be threaded on to a ring, is:

(*a*) 672 (*b*) 1344 (*c*) 336 (*d*) 1433

17. How many ways are there to seat n married couples ($n \geq 3$) around a table such that men and woman alternate and each woman is not adjacent to her husband.

(*a*) $n!(n-1)!-2(n-1)!$ (*b*) $2(n-1)!-n!(n-1)!$

(*c*) $n!(n-1)!$ (*d*) $2n!(n-1)!$

18. The number of ways in which 6 men and 5 woman can dine at a round table if no two woman are to sit together is given by

(*a*) $6!\times 5!$ (*b*) 6×5 (*c*) 30 (*d*) 3×4

COMBINATION AND IT'S GEOMETRIAL APPLICATION

19. Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels?

(*a*) 120 (*b*) 25200 (*c*) 4200 (*d*) 42000

20. In a place these are 37 straight lines of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Then the number of intersection points the lines have is equal to

(*a*) 535 (*b*) 601 (*c*) 728 (*d*) 730

21. From six gentlemen and four ladies, a committee of five is to be formed. Number of ways in which this can be done if the committee is to include at least one lady is

(*a*) 252 (*b*) 246 (*c*) 248 (*d*) 250

22. Which of the following is correct

(*a*) The number of diagonals that can be drawn by joining the vertices of an octagon is 28.

(*b*) If a polygon has 54-diagonals, the number of its sides is 9.

(*c*) In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon be 70 then the number of diagonals of the polygon is 20.

(*d*) All above are correct.

23. The number of ways in which one of more balls can be selected out of 10 white, 9 green and 7 blue balls is

(*a*) 892 (*b*) 881 (*c*) 891 (*d*) 879

24. A lady gives a dinner party to 5 guests to be selected from nine friends. The number of ways of forming the party of 5, given that two of the friends will not attend the party together is

(*a*) 56 (*b*) 126 (*c*) 91 (*d*) 92

25. The number of words of 5 letters that can be made with the letters of the word "PROPOSITION'.

(*a*) 6890 (*b*) 7000 (*c*) 6800 (*d*) 6900

DIVISORS OF A GIVEN NATURAL NUMBER

26. The number of proper divisors of $a^p b^q c^r d^s$ where a, b, c, d are primes & $p, q, r, s \in N$, is

(*a*) $pqrs$

(*b*) $(p+1)(q+1)(r+1)(s+1)-4$

(*c*) $pqrs-2$

(*d*) $(p+1)(q+1)(r+1)(s+1)-2$

Credit:--- Divansh.....

27. N is a least natural number having 24 divisors. Then the number of ways N can be resolved into two factors is

(*a*) 12 (*b*) 24 (*c*) 6 (*d*) 18

28. How many divisors of 21600 are divisible by 10 but not by 15?

(*a*) 10 (*b*) 30 (*c*) 40 (*d*) 20

GROUPING

29. A team of 8 students goes on an excursion, in two cars, of which one can seat 5 and the other only 4. If internal arrangement inside the car does not matter then the number of ways in which they can travel, is

(*a*) 91 (*b*) 182 (*c*) 126 (*d*) 3920

30. 9 students of class 12th of whom 2 from section A, 3 from section B and 4 from section C are available for inter school debate competition. Number of ways in which 4 students can be selected so that there is atleast 1 from each section, is

(*a*) 144 (*b*) 120 (*c*) 72 (*d*) 71

31. There are five different peaches and three different apples. Number of ways they can be divided into two packs of four fruits if each pack must contain atleast one apple, is

(*a*) 95 (*b*) 65 (*c*) 60 (*d*) 30

MULTINOMIAL THEOREM

32. The number of ways in which 10 identical apples can be distributed among 6 children so that each child receives atleast one apple is:

(*a*) 126 (*b*) 252 (*c*) 378 (*d*) 152

33. The number of terms in a complete homogeneous expression of degree n in x, y and z is

(*a*) n (*b*) $\frac{n(n+1)}{2}$

(*c*) $\frac{(n+1)(n+2)}{2}$ (*d*) $\frac{n(n-1)}{2}$

34. Number of ways of selecting 5 coins from coins three each of Rs. 1, Rs. 2 and Rs. 5 if coins of the same denomination are alike, is:

(*a*) 9 (*b*) 12 (*c*) 21 (*d*) 15

35. The number of ways in which 16 identical things can be distributed among 4 persons if each person gets at least 3 things, is

(*a*) 33 (*b*) 35 (*c*) 38 (*d*) 36

DERANGEMENTS AND MISCELLENOUS

36. The sum of all the numbers which can be formed by using the digits 1, 3, 5, 7 all at a time and which have no digit repeated, is

(*a*) $16 \times 4!$ (*b*) $1111 \times 3!$

(*c*) $16 \times 1111 \times 3!$ (*d*) $16 \times 1111 \times 4!$

37. Number of numbers greater than a million using the digits 2, 3, 0, 3, 4, 2, 5 of these how many are div. by 5.

(*a*) 330, 1080 (*b*) 1080, 330

(*c*) 1000, 300 (*d*) 300, 1000

38. In case these 6 toys are to be distributed equally between R/S/G then our answer will be

(*a*) 75 (*b*) 90 (*c*) 100 (*d*) 80

39. Number of ways in which 12 different books can be tied up in 3 bundles one containing 2 and the other two bundles containing 5 different books.

(*a*) $\frac{12!}{5!5!2!2!}$ (*b*) $\frac{12!}{5!2!}$

(*c*) $\frac{12!}{5!5!2!}$ (*d*) None of these

40. Maximum number of points of intersection of six circles in the plane

(*a*) 20 (*b*) 25 (*c*) 30 (*d*) 35

41. Nine hundred distinct N-digit numbers are to be formed by using 6, 8 and 9 only. The smallest value of N for which this is possible, is

(*a*) 6 (*b*) 7 (*c*) 8 (*d*) 9

42. Number of distinct terms in the expansions of $(x+y+z+w)^{50}$ is equal to.

(*a*) ${}^{53}C_3$ (*b*) 51 (*c*) 51^2 (*d*) ${}^{51}C_3$

43. Number of natural solution of the equation $x+y+z=102$.

(*a*) 5050 (*b*) 4050

(*c*) 4950 (*d*) None of these

44. Two American, Two British, Two Chinese and 4 are of different countries can be seated if persons of the same nationality are to be separated.

(*a*) $47 \times 8!$ (*b*) $48 \times 8!$

(*c*) $43 \times 9!$ (*d*) None of these

45. A set contains $(2n+1)$ elements. If the number of subsets of this which contain at most n elements is 4096, then the value of n is-

(*a*) 6 (*b*) 15 (*c*) 21 (*d*) 12

46. In how many ways a garland can be made from exactly 10 flowers?

(*a*) 10! (*b*) 9! (*c*) 2(9!) (*d*) $\frac{9!}{2}$

47. Number of divisors of $n = 38808$ (except 1 and n) is

(*a*) 70 (*b*) 68 (*c*) 72 (*d*) 74

48. The number of positive integral solutions of $abc = 30$ is

(*a*) 30 (*b*) 27 (*c*) 8 (*d*) 16

49. In how many ways 3 letters can be posted in 4 letter-boxes, if all the letters are not posted in the same letter-box

(*a*) 63 (*b*) 60 (*c*) 77 (*d*) 81

50. The exponent of 3 in 100! is

(*a*) 33 (*b*) 44 (*c*) 48 (*d*) 52

Prabal (JEE Main Level)

1. The number of 4 digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contain digit 1 is (Repetition is not allowed)

(a) 1225 (b) 1252 (c) 750 (d) 480

2. The number of word groups by taking at least 1 letters of each words 'PATH', 'GROW' and 'SKIN' are-

(a) $15^3 - 1$ (b) 15^3 (c) $16^3 - 1$ (d) 16^3

3. The total number of seven digit numbers the sum of whose digits is even is-

(a) 9000000 (b) 4500000 (c) 8100000 (d) 450000

4. ${}^1P_1 + 2\cdot{}^2P_2 + 3\cdot{}^3P_3 + \ldots.. + n\cdot{}^nP_n =$

(a) ${}^{n+1}P_{n+1}$ (b) ${}^{n+1}P_{n+1} - 1$
(c) ${}^{n+1}P_{n+1} - 2$ (d) ${}^{n+1}P_n$

5. There are 9 children comprising of 3 boys and 6 girls. Let m denotes the number of ways when the 3 boys are in succession and n denotes the corresponding figure when the two end position are occupied by boys. If $m = kn$ then k is equal to

(a) 3 (b) 2 (c) 1 (d) (1/3)

6. In a football championship, there were played 153 matches. Every team played one match with each other. The number of teams participating in the championship is

(a) 17 (b) 18 (c) 9 (d) 13

7. In how many ways a team of 11 players can be formed out of 25 players, if 6 out of them are always to be included and 5 are always to be excluded

(a) 2020 (b) 2002 (c) 2008 (d) 8002

8. In how many ways can a committee consisting of one or more members be formed out of 12 members of the Municipal Corporation

(a) 4095 (b) 5095 (c) 4905 (d) 4090

9. The number of ways in which we can select three numbers from 1 to 30 so as to exclude every selection of all even numbers is

(a) 4060 (b) 3605 (c) 455 (d) 446

10. The number of numbers between 1 and 10^{10} which contain the digit 1 is-

(a) $10^{10} - 9^{10} - 1$ (b) 9^{10}
(c) $10^{10} - 8^{10}$ (d) 9^9

11. Number of ways of selection of 4 letters of the word "PERFECT" is

(a) 15 (b) 21 (c) 25 (d) 31

12. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

(a) 5 (b) 7 (c) 6 (d) 4

13. Between two junction stations A and B there are 12 intermediate stations. The number of ways in which a train can be made to stop at 4 of these stations so that no two of these halting stations are consecutive is-

(a) ${}^8C_4$ (b) ${}^9C_4$
(c) ${}^{12}C_4 - 4$ (d) ${}^9C_3$

14. The number of ways of selecting n things out of $3n$ things of which n are of one kind and n are of a second kind and alike and the rest are unlike, is.

(a) $(n+2)^{2n}$ (b) $(n+2)2^{n-1}$
(c) $(n+2)2^{n+1}$ (d) $n + 2^n$

15. To fill 12 vacancies there are 25 candidates of which five are from scheduled caste. If 3 of the vacancies are reserved for scheduled caste candidates while the rest are open to all, then the number of ways in which the selection can be made

(a) ${}^{20}C_7 + 5\times{}^{20}C_8 + 10\times{}^{20}C_9$

(b) ${}^{22}C_9 - {}^5C_3$

(c) ${}^{22}C_3 + {}^5C_3$

(d) ${}^5C_3 - {}^{22}C_9$

16. All possible 120 permutations of WDSMC are arranged in dictionary order, as if each were an ordinary five-letter word. The last letter of the 86th word in the list, is

(a) W (b) D (c) M (d) C

17. The number of ways in which 5 male and 2 female members of a committee can be seated around a round table so that the two female are not seated together is

(a) 480 (b) 600 (c) 720 (d) 840

18. The number of ways in which four letters of the word 'MATHEMATICS' can be arranged is given by

(a) 136 (b) 192 (c) 1680 (d) 2454

19. The total number of ways of selecting six coins out of 20 one rupee coins, 10 fifty paise coins and 7 twenty five paise coins is

(a) 28 (b) 56 (c) ${}^{37}C_6$ (d) ${}^{37}P_6$

20. The number of six digit numbers between 700000 and 800000 such that the first digit is equal to the sum of the other five digits is

(a) 1512 (b) 330 (c) 290 (d) 130

21. In an eleven storey building 9 people enter a lift cabin. It is known that they will leave the lift in groups of 2, 3 and 4 at different residential storeys. Number of ways in which they can get down is:

(a) $\dfrac{(12)!}{2!(3!)^2.4!}$ (b) $\dfrac{10!}{4}$

(c) $\dfrac{10!}{4!}$ (d) $\dfrac{(12)!}{(3!)^2\cdot4!}$

22. Sixteen players $S_1, S_2, S_3, \ldots.., S_{16}$ play in a tournament. Number of ways in which they can be grouped into eight pairs so that S_1 and S_2 are in different groups, is equal to

Credit:--- Divansh.....

(a) $\frac{(14)!}{2^6 \cdot 6!}$ (b) $\frac{(15)!}{2^7 \cdot 7!}$

(c) $\frac{(14)!}{2^7 \cdot 6!}$ (d) $\frac{(14)!}{2^6 \cdot 7!}$

23. If $A_1A_2 \ldots A_{18}$ is a regular polygon, then number of ways in which a quadrilateral can be formed with the help of vertices of given polygon, such that no side of quadrilateral is a side of given polygon,

(a) 1365 (b) 1287 (c) 3060 (d) 715

24. The number of odd proper divisors of $3^p . 6^m . 21^n$ is

(a) $(p+1)(m+1)(n+1)-2$

(b) $(p+m+n+1)(n+1)-1$

(c) $(p+1)(m+1)(n+1)-1$

(d) $(p+1)(m+1)(n+1)$

25. Let there be $n \geq 3$ circles in a plane. The value of n for which the number of radical centres is equal to the number of radical axes is (assume that all radical axes and radical centre exist and are different)

(a) 7 (b) 6

(c) 5 (d) None of these

26. If a cricket team of 11 players is to be selected from 8 bastman, 6 bowlers , 4 all rounder and 2 wicket keepers, then number of selection when 2 particular bastmen don't want to play when a particular bowler will play is

(a) ${}^{17}C_{10} + {}^{19}C_{11}$ (b) ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$

(c) ${}^{17}C_{10} + {}^{20}C_{11}$ (d) ${}^{19}C_{10} + {}^{19}C_{11}$

27. Number of words can be formed using all the letters of the work 'HIPHIPHURRAY' In which all H's lie somewhere between R's is

(a) (198)7! (b) (99)7! (c) (99)8! (d) (198)8!

28. Let $A = \{1, 2, 3, 4\}$. The number of different ordered pairs (B, C) that can be formed such that can be formed such that $B \subseteq A, C \subseteq A$ and $B \cap C = \phi$

(a) 3 (b) 9 (c) 27 (d) 81

29. There were two women participating in a chess tournament. Every participant played two games with every other participant. The number of games that the men played among themselves exceeds by 66 that of the games which the man played with the two women, then total number of participants is

(a) 11 (b) 13 (c) 10 (d) 12

30. Rajdhani express stops at six intermediate stations between Kota and Mumbai. Five passengers board at Kota. Each passenger can get down at any station till Mumbai. Then find number of ways so that all five passengers will get down at different stations, is

(a) ${}^6P_5$ (b) ${}^6C_5$ (c) ${}^7P_5$ (d) ${}^7C_5$

31. A bag contains 17 markers with numbers 1 to 17. A marker is drawn at random and then replaced; a second marker is drawn then find number of ways so that first number is even and second is odd, is

(a) 36 (b) 128 (c) 256 (d) 72

32. If all numbers starting from 1 are listed in increasing order by using the digits 0, 1, 2, 3, 4, 5, 6 then the number whose rank is 8000 is

(a) 32605 (b) 32116 (c) 32216 (d) 33316

33. A cricket team comprising of A, B, C, ..., J, K is to be sent for batting. If 'A' wants to bat before 'J' and 'J' wants to bat after 'G'. Then the number of batting orders if other players could go in any order.

(a) ${}^{11}C_2 2.8!$ (b) ${}^{11}C_3 2.8!$

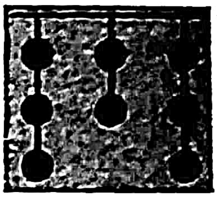
(c) ${}^{11}C_4 2.8!$ (d) ${}^{11}C_5 2.8!$

34. Rajdhani express bound from Delhi to Bombay stops at 9 intermediate stations. 6 passengers enter the train during the journey holding 6 different tickets of the same class (K classes). How many different sets of tickets they may have had.

(a) ${}^{47k}C_6$ (b) ${}^{43k}C_6$ (c) ${}^{45k}C_6$ (d) ${}^{46k}C_6$

INTEGER TYPE QUESTIONS

35. 8 clay targets are arranged as shown. If N be the number of different ways they can be shot (one at a time) if no target can be shot until the target(s) below it have been shot. Find the ten's digit of N.



36. Find number of positive integral solutions satisfying the equation $(x_1 + x_2 + x_3)(y_1 + y_2) = 77$.

37. Number of five digits numbers divisible by 3 that can be formed using the digits 0, 1, 2, 3, 4, 7 and 8 if, each digit is to be used atmost one is N where 'N' is three digit number then sum of the digit of 'N'

38. Find number of positive integral solutions satisfying the equation $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 = 240$.

39. Sum of all the numbers that can be formed using all the digits 2, 3, 3, 4, 4, 4, is N then sum of the digit of the number 'N' equals

40. The number of polynomials of the form $x^3 \div ax^2 \div bx \div c$ which are divisible by $x^2 + 1$ where $a, b, c \in \{1, 2, 3, \ldots, 10\}$ is $10K$, then K is ______.

41. Number of numbers greater than 1000 from the digits 1, 1, 2, 3, 4, 0 taken 4 at a time.

42. How many 8 digit numbers can be formed using two 1's, two 2's, two 3's, one 4 and one 5 so that no two consecutive digits are identical.

43. The total number of five-digit numbers of different digits in which the digit in the middle is the largest is

44. Maximum number of points of intersection of six parabolas

45. The number of odd proper divisor of 24300 is $N \div 11$, then N is equal to ______.

46. There are 15 seats in the first row of a theatre of which 6 are to be occupied. The number of ways of arranging 6 person so that no two persons are side by side is $k \div 5$, then digit in unit's place in k is equal to

47. Number of different collections of 7 letters that can be formed from the letters of the word ACCOMMODATION.

48. How many other words can be formed using the letters of the word "CIVILISATION" without changing the relative order of vowels and consonants.

Parikshit (JEE Advanced Level)

SINGLE CORRECT TYPE QUESTIONS

1. A condolence meeting is being held in a hall which has 7 doors, by which mourners enter the hall. One can use any of the 7 doors to enter and can come at any time during the meeting. At each door, a register is kept in which a mourner has to affix his signature while entering the hall. If 200 people attend the meeting, how many different sequences of 7 lists of signatures can arise?

   (a) $\frac{206!}{6!}$ (b) $\frac{206!}{3!}$ (c) $\frac{206!}{5!}$ (d) $\frac{206!}{4!}$

2. There are p intermediate railway stations on a railway line from one terminal to another. In how many ways can a train stop at three of these intermediate stations if no two of these stations (where it stops) are to be consecutive?

   (a) ${}^{p+2}C_3$ (b) ${}^{p-2}C_3$ (c) ${}^{p-1}C_3$ (d) ${}^{p-3}C_3$

3. Number of ways in which $5B$ and $3G$ can be seated on a circle if a particular B_1 and G_1 are never adjacent to each other in any arrangement.

   (a) $[5 \times 6!]$ (b) $[2 \times 6!]$ (c) $[4 \times 6!]$ (d) $[3 \times 6!]$

4. Let $x_1, x_2, ..., x_k$ be the divisors of positive integer 'n' (including 1 and n). If $x_1 + x_2 + ... + x_k = 75$, then $\sum_{i=1}^{k} \frac{1}{x_i}$ is equal to

   (a) $\left[\frac{75}{n}\right]$ (b) $\left[\frac{73}{n}\right]$ (c) $\left[\frac{76}{n}\right]$ (d) $\left[\frac{79}{n}\right]$

5. The total number of ways in which n^2 number of identical balls can be put in n numbered boxes (1, 2, 3, ..., n) such that i^{th} box contains at least i number of balls is

   (a) ${}^{\left(\frac{n^2+n+2}{2}\right)}C_{n-1}$ (b) ${}^{\left(\frac{n^2+n-2}{2}\right)}C_{n-1}$

   (c) $\left(\frac{n^2-n+2}{2}\right){}^{n}C_{n-1}$ (d) None of these

6. If three one by one squres are selected at random from the chessboard, then the probability that they form the letter 'L' is

   (a) $\frac{196}{{}^{64}C_3}$ (b) $\frac{49}{{}^{64}C_3}$

   (c) $\frac{36}{{}^{64}C_3}$ (d) $\frac{98}{{}^{64}C_3}$

7. Two teams are to play a series of five matches between them. A match ends in a win, loss or draw for a team. A number of people forecast the result of each match and no two people make the same forecast for the series of matches. The smallest group of people in which one person forecasts correctly for all the matches will contain n people, where n is

   (a) 81 (b) 243

   (c) 486 (d) None of these

8. In a certain test, there are n questions. In the test 2^{n-i} students gave wrong answers to at least i questions, where $i = 1, 2, ..., n$. If the total number of wrong answers given is 2047, then n is equal to

   (a) 10 (b) 11 (c) 12 (d) 13

9. n is selected from the set {1, 2, 3, ..., 100) and the number $2^n + 3^n + 5^n$ is formed. Total number of ways of selecting n so that the formed number is divisible by 4 is equal to

   (a) 50 (b) 49

   (c) 48 (d) None of these

10. The number 916238457 is an example of nine digit number which contain each of the digits 1 to 9 exactly once, it also has the property that the digit 1 to 5 occur in their nautral order while the digit 1 to 6 do not. Number of such numbers are

    (a) 2268 (b) 2520 (c) 2975 (d) 1560

11. Consider a set {1, 2, 3, ..., 100). The number of ways in which a number can be selected from the set so that it is of the form x^y, (where $x, y \in N$ and ≥ 2), is

    (a) 12 (b) 16 (c) 5 (d) 11

MULTIPLE CORRECT TYPE QUESTIONS

12. Consider a dodecagon (12 sides) then:

    (a) Number of ways of selecting two triangles which can be made using the vertices of dodecagon so that they do not have a common side is 14692

    (b) Number of ways of selecting two triangle which can be made using the vertices of dodecagon so that they do not have a common side is 21120

    (c) Number of hexagons which can be formed using the 12 vertices such that none of sides of hexagon is also a side of dodecagon is 4

    (d) Assume the dodecagon to be regular and also consider its centre then the number of straight line formed using 13 vertices is 66

13. Which of the following statements are correct?

    (a) Number of words that can be formed with 6 of the letters of the word "CENTRIFUGAL" if each word must contain all the vowels is $3 \cdot 7!$

    (b) There are 15 balls of which some are white and the rest black. If the number of ways in which the balls can be arranged in a row, is maximum then the number of white balls must be equal to 7 or 8. Assume balls of the same colour to be alike.

    (c) There are 12 things, 4 alike of one kind, 5 alike of another kind and the rest are all different. The total number of combinations is 240.

    (d) Number of selections that can be made of 6 letters from the word "COMMITTEE" is 35.

Credit:--- Divansh.....

14. The sequence of all positive palindromes are written in ascending order 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22,...

Then

(*a*) The 2018 positive palindrome is 1019101

(*b*) The 1998 positive palindrome is 1001001

(*c*) The rank of 2140412 is 3139

(*d*) The rank of 1999991 is 2998

15. Consider the word D = F R E E W H E E L. Which of the following statement(s) is/are correct

(*a*) Number of other ways in which the letters of the word D can be arranged is ${}^9P_5$.

(*b*) Number of ways is which the letters of the word *D* can be an arranged in a circle is ${}^8P_4$ distisguishing between clockwise and anticlockwise.

(*c*) Number of ways in which the letters of the word *D* can be arranged if vowels and consonants both are in alphabetical order is $\frac{1}{2}\,{}^{10}C_5$

(*d*) If as many more words are formed as possible using the letters of the word *D* the number of words which contain the word FEEL is ${}^6P_4$.

16. The number of ways of arranging the letters *AAAAA*, *BBB*, *CCC*, *D*, *EE* & *F* in a row if the letter *C* are separated from one another is:

(*a*) ${}^{13}C_3\cdot\frac{12!}{5!\,3!\,2!}$ (*b*) $\frac{13\,!}{5\,!\;3\,!\;3\,!\;2\,!}$

(*c*) $\frac{14\,!}{3\,!\;3\,!\;2\,!}$ (*d*) $11\cdot\frac{13\,!}{6\,!}$

17. Number of ways in which the letters of the word "TOMATO" can be arranged if two alike vowels are separated, is not equal to

(*a*) Number of ways in which letters of the word "RAITHATHA" can be arranged if no two alike letters are separated.

(*b*) Number of 3 digit numbers with atleast one 3 and atleast one 2.

(*c*) Number of ways in which Ram and Rama can exchange their maps if Ram has 3 and Rama has 7 maps, all maps being different, maintaining their original number of maps at the end.

(*d*) Number of ways in which 2 alike apples and 4 alike oranges can be distributed in three children if each child get none, one or more fruits.

18. A forecast is to be made of the results of five cricket matches, each of which can be a win, draw or a loss for Indian team. If *p* denotes the number of forecast with exactly 1 error, *q* denotes the number of forecast with exactly 3 error and '*r*' denotes the number of forecast with all error, then the correct statements are:

(*a*) $2q = 5r$ (*b*) $8p = q$

(*c*) $8p = 5r$ (*d*) $2(p + r) > q$

19. Coefficient of $x^2 y^3 z^4$ in the expansion of $(x + y + z)^9$ is equal to

(*a*) The number of ways in which 9 things of which 2 alike of one kind, 3 alike of 2nd kind, and 4 alike of 3rd kind can be arranged.

(*b*) The number of ways in which 9 identical things can be distributed in 3 persons each receiving atleast two things.

(*c*) The number of ways in which 9 identical things can be distributed in 3 persons each receiving none one or more.

(*d*) The number of ways in which 9 different books can be tied up in to three bundles one containing 2, other 3 and third containing 4 books.

20. The number of ways of choosing triplet (x, y, z) such that $z > \max\{x, y\}$ and $x, y, z \in \{1, 2, ..., n, n+1\}$ is

(*a*) ${}^{n+1}C_3 + {}^{n+2}C_3$

(*b*) ${}^{n+1}C_2 + 2({}^{n+1}C_3)$

(*c*) $1^2 + 2^2 + ... + n^2$

(*d*) $2({}^{n+2}C_3) - {}^{n+1}C_2$

21. Number of ways in which 30 identical things are distributed among 6 persons is

(*a*) ${}^{17}C_5$ if each gets odd number of things

(*b*) ${}^{16}C_{11}$ if each gets odd number of things

(*c*) ${}^{14}C_5$ things if each gets even number of things (excluding 0)

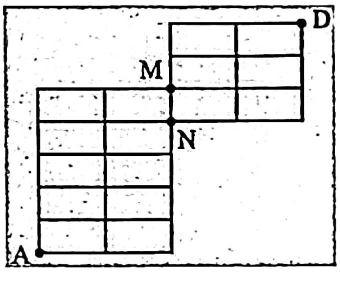
(*d*) ${}^{15}C_{10}$ if each gets even number of things (excluding 0)

22. Number of ways in which three numbers in A.P. can be selected from 1, 2, 3, ..., *n* is

(*a*) $\left(\frac{n-1}{2}\right)^2$ if *n* is even (*b*) $\frac{n(n-2)}{4}$ if *n* is even

(*c*) $\frac{(n-1)^2}{4}$ if *n* is odd (*d*) $\frac{(n-1)^2}{2}$ if *n* is odd

23. Let a person has to go from A to D moving along horizontal and vertical grids using shortest path (as shown in figure). Then



(*a*) The number of paths is equal to 276.

(*b*) The number of paths is equal to 186.

(*c*) The number of paths in which he doesn't cross point N is equal to 36.

(*d*) The number of paths in which he doesn't cross point M is equal to 60.

Credit:--- Divansh.....

24. Number of ways in which 6 different toys can be distributed equally among 3 children is also equal to

(*a*) number of ways in which all the letters of the word KUMKUM can be arranged in a definite order.

(*b*) number of different signals which can be obtained from 10 different flags taken two at a time keeping one above the other.

(*c*) number of ways in which five different marbles can be placed in three different boxes so that no box remains empty.

(*d*) number of ways in which all the letters of the word CALCULUS can be arranged without changing the order and position of non alike letters.

25. There are 10 seats in the first row of a theatre of which 4 are to be occupied. The number of ways of arranging 4 persons so that no two persons sit side by side is

(*a*) $^7C_4$ (*b*) $4 \cdot {}^7C_3$

(*c*) $^7C_3 \times 4!$ (*d*) 840

26. Let $S = x_1 + x_2 + x_3 + x_4 + x_5$ and $K = x_1 + x_2 + x_3$ then

(*a*) Number of positive integral solutions of the equation $S = 21$ is 4845

(*b*) Number of integral solutions of the equation $S = 21$ such that $x_i \geq 2\ \forall_i = 1, 2, 3, 4, 5$ is 1365.

(*c*) Number of integral solutions of the equation $S = 20$ is 42404

(*d*) Number of integral solutions $(x_1, x_2, x_3, x_4, x_5)$ such that $S = 20$ and $K = 5$ is 336

COMPREHENSION BASED QUESTIONS

Comprehension-1 (Qs. 27 and 28): Consider a polygon of sides 'n' which satisfies the equation, $3 \cdot {}^nP_4 = {}^{n-1}P_5$

27. Number of quadrilaterals that can be made using the vertices of the polygon of sides 'n' if exactly two adjacent sides of the quadrilateral are common to the sides of the polygon, is

(*a*) 50 (*b*) 60

(*c*) 70 (*d*) 84

28. Number of quadrilaterals that can be formed using the vertices of a polygon of sides 'n' if exactly 1 side of the quadrilateral is common with the side of the polygon, is

(*a*) 150 (*b*) 100

(*c*) 96 (*d*) 104

Comprehension-2 (Qs. 29 to 30): When letters of any word are written in all possible ways using all letters of given word and these words are arranged as in a dictionary then the position of the word in that list in called the rank of the word.

29. What is the rank of word 'PARALLEL

(*a*) 2353 (*b*) 2629

(*c*) 2593 (*d*) 2623

30. If list of all words is made by using the letters of word IITJEE in dictionary order then the word at 14 position is

(*a*) JEE IIT (*b*) JII TEE

(*c*) JIE TIE (*d*) JII EE

Comprehension-3 (Qs. 31 to 33): In how many ways can letters of the word ARRANGE be arranged so that

31. The two R' are never together

(*a*) 900 (*b*) 872

(*c*) 960 (*d*) 980

32. The two A's are together but not the two R's are together

(*a*) 240 (*b*) 260

(*c*) 300 (*d*) 340

33. Neither two A's nor the two R's are together.

(*a*) 660 (*b*) 600

(*c*) 640 (*d*) 690

Comprehension-4 (Qs. 34 to 36): We have to choose 11 players for cricket team from 8 batsmen, 6 bowlers, 4 all-rounders, and 2 wicket keepers.

34. The number of selections when almost one all-rounder and one wicket keeper will play is

(*a*) $^4C_1 \times {}^{14}C_{10} + {}^2C_1 \times {}^{14}C_{10} + {}^4C_1 \times {}^2C_1 \times {}^{14}C_9 + {}^{14}C_{11}$

(*b*) $^4C_1 \times {}^{15}C_{11} + {}^{15}C_{11}$

(*c*) $^4C_1 \times 15C_{10} + {}^{15}C_{11}$

(*d*) $^4C_1 \times {}^{15}C_9 + {}^{15}C_{10}$

35. The number of selections when two particular batsmen do not want to play when a particular bowler will play is

(*a*) $^{17}C_{10} + {}^{19}C_{11}$

(*b*) $^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$

(*c*) $^{17}C_{10} + {}^{20}C_{11}$

(*d*) $^{19}C_{10} + {}^{19}C_{11}$

36. The number of selection when a particular batsman and a particular wicket keeper do not want to play together is

(*a*) $2{}^{18}C_{10}$

(*b*) $^{19}C_{11} + {}^{18}C_{10}$

(*c*) $^{19}C_{10} + {}^{19}C_{11}$

(*d*) $^{19}C_6 + {}^{18}C_9$

Comprehension-5 (Qs. 37 to 38): A is a set containing n elements A subset S_1 of A is chosen. The set A is reconstructed by replacing the elements of S_1. Again, a subset S of A is chosen and again the set is reconstructed by replacing the elements of S_2. The number of ways of choosing S_1 or S_2 where

37. S_1 and S_2 have one element common is

(*a*) 3^{n-1} (*b*) $n \cdot 3^{n-1}$

(*c*) 2^{n-1} (*d*) n

38. $S_1 \cup S_2 = A$ is

(*a*) 3^n (*b*) $n \cdot 3^n$

(*c*) 4^n (*d*) 4^{n-1}

MATCH THE COLUMN TYPE QUESTIONS

39. Match the column

	Column – I		Column – II
A.	The total number of selections of atleast one fruit which can be made from, 3 bananas, 4 apples and 2 oranges is, it is given that fruits of one kind are identical.	p.	120
B.	There are 10 true-false statements in a question paper. How many sequences of answers are possible in which exactly three are correct?	q.	286
C.	The number of ways of selecting 10 balls from unlimited number of red, black, white and green balls is, it is given that balls of same colours are identical.	r.	59
D.	The number of words which can be made from the letters of the word 'MATHEMATICS' so that consonants occur together?	s.	75600

(*a*) A → r; B → p; C → q; D → s

(*b*) A → s; B → p; C → q; D → r

(*c*) A → r; B → s; C → q; D → p

(*d*) A → s; B → q; C → p; D → r

40. PHYSICS WALLAH, Total 12 letters. Set of repeating letters = {H, A, L}

	Column -I		Column-II
A.	The two A's are not together	p.	$\frac{5}{4}(11!)$
B.	Neither two A's nor two L are together	q.	$57\frac{10!}{2!2!}$
C.	The number of words taking all letters of the given word such that atleast one repeating letter is at odd position is	r.	$\frac{12!}{2!2!2!}\left[1-\frac{6!\times 6!}{12!}\right]$ *OR* 83070×61
D.	The number of words formed taking all letters of the given word such that in each word both H are together and both L's are together but both A's are not together is	s.	$4\times 9!$

(*a*) A → r; B → q; C → p; D → s

(*b*) A → p; B → q; C → s; D → r

(*c*) A → p; B → q; C → r; D → s

(*d*) A → p; B → r; C → q; D → s

41. Match the column

	Column – I		Column – II
A.	There are 12 points in a plane of which 5 are collinear. The number of distinct convex quadrilaterals which can be formed with vertices at these points is:	p.	185
B.	If 7 points out of 12 are in the same straight line, then the number of triangles formed is	q.	420
C.	If AB and AC be two line segemets and there are 5, 4 points on AB and AC (other than A), then the number of quadrilateral, with vertices on these points equals	r.	126
D.	The maximum number of points of intersection of 8 unequal circles and 4 straight lines.	s.	60

(*a*) A → r; B → s; C → p; D → q

(*b*) A → q; B → p; C → s; D → r

(*c*) A → q; B → s; C → p; D → r

(*d*) A → p; B → q; C → r; D → s

42. Match the following:

	Column – I		Column – II
A.	If n be the numbers between 500 and 4000 can be formed with the digits 2, 3, 4, 5, 6 when repetition is not allowed, then n is divisible by	p.	2
B.	If n be the number of even numbers between 200 and 3000 can be formed with the digit 0, 1, 2, 3, 4 when repetition is not allowed, then n is divisible by	q.	3
C.	If n be the number of words that can be made by arranging the letters of the word ROORKEE that neither begin with R nor end with E, then n is divisible by	r.	5
D.	In a class tournament when the participants where to play one game with another, two class players fell ill, having played 3 games each. If the total number of games played is 84, the number of participants at the beginning was k then k is divisible by	s.	11
		t.	17

(*a*) A → q,r; B → q; C → p,q,r,s; D → p,q

(*b*) A → p,q; B → q,r,s; C → p; D → q,r

(*c*) A → p,q,r,s; B → q; C → p,q; D → q,r

(*d*) A → p,q; B → q; C → p,q,r,s; D → q,r

Credit:--- Divansh.....

43. Match the following Column-I with Column-II.

	Column-I		Column-II
A.	Four dice (six faced) are rolled. The number of possible outcomes in which at least one dice shows 2 is	p.	210
B.	Let A be the set of 4-digit number $a_1a_2a_3a_4$ where $a_1 > a_2 > a_3 > a_4$. Then $n(A)$ is equal to	q.	750
C.	The total number of three-digit numbers, the sum of whose digits is even, is equal to	r.	671
D.	The number of four-digit numbers without repetition that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contains digit 1 is	s.	450

(*a*) A → s; B → q; C → p; D → r

(*b*) A → r; B → p; C → s; D → q

(*c*) A → p; B → r; C → q; D → s

(*d*) A → r; B → s; C → p; D → q

INTEGER TYPE QUESTIONS

44. The number of ways to choose 7 distinct natural numbers from the first 100 natural numbers such that any two chosen numbers differ atleast by 7 can be expressed as $^nC_7$. Find the number of divisors of n.

45. A box contains 6 balls which may be all of different colours or three each of two colours or two each of three different colours. The number of ways of selecting 3 balls from the box (if ball of same colour are identical), is N then sum of the digits in the number 'N' equals

46. Six married couple are sitting in a room. Number of ways in which 4 people can be selected so that there is exactly one married couple among the four is N then number of divisors of N equals.

47. How many hexagons can be constructed by joining the vertices of a quindecagon (15 sides) if none of the sides of the hexagon is also the side of the 15-gon.

48. If number of 7 digit numbers if the sum of their digits is 59 is α and out of these are divisible by 11 (Digits to be used can be 5, 6, 7, 8, 9) is β, then find $\alpha + \beta$.

49. How many different words can be formed using all the letters of the words HONOLULU if (no two alike letters are together).

50. A shelf contains 6 separate compartments. Number of ways in which 12 indistinguishable (like) marbles can be placed in the compartments so that no compartment is empty.

51. The number of five digit numbers that can be formed using its digits 1, 2, 3, 4, 5, 6, 7, 8, 9 in which one digit appears once and two digits appear twice (e.g. 44533) is k then the value of $k/1890$ is equal to ____.

PYQ's (Past Year Questions)

FUNDAMENTAL PRINCIPLE OF COUNTING, COMBINATION, ARRANGEMENT

1. 60 words can be made using all the letters of the word BHBJO, with or without meaning. If these words are written as in a dictionary, then the 50th word is : [05 April, 2024 (Shift-II)]

(*a*) OBBHJ (*b*) HBBJO

(*c*) OBBJH (*d*) JBBOH

2. If all the words with or without meaning made using all the letters of the word "NAGPUR" are arranged as in a dictionary, then the word at 315th position in this arrangement is: [06 April, 2024 (Shift-II)]

(*a*) NRAGUP (*b*) NRAGPU

(*c*) NRAPGU (*d*) NRAPUG

3. The number of five digit numbers, greater than 40000 and divisible by 5, which can be formed using the digits 0, 1, 3, 5, 7 and 9 without repetition, is equal to [12 April, 2023 (Shift-I)]

(*a*) 120 (*b*) 132

(*c*) 72 (*d*) 96

4. The number of 4-letter words, with or without meaning, each consisting of 2 vowels and 2 consonants, which can be formed from the letters of the word UNIVERSE without repetition is [06 April, 2023 (Shift-II)]

5. The number of ways to distribute 30 identical candies among four children C_1, C_2, C_3, and C_4, so that C_2, receives atleast 4 and atmost 7 candies, C_3, receives atleast 2 and atmost 6 candies, is equal to : [28 June, 2022 (Shift-II)]

(*a*) 205 (*b*) 615 (*c*) 510 (*d*) 430

6. The number of 4-digit integers in the closed interval [2022, 4482] formed by using the digits 0, 2, 3, 4, 6, 7 is ______ [JEE Adv, 2022]

7. A scientific committee is to be formed from 6 Indians and 8 foreigners, which includes at least 2 Indians and double the number of foreigners as Indians. Then the number of ways the committee can be formed is: [24 Feb, 2021 (Shift-I)]

(*a*) 560 (*b*) 1625 (*c*) 1050 (*d*) 575

Credit:--- Divansh.....

8. If the digits are not allowed to repeat in any number formed by using the digits 0, 2, 4, 6, 8, then the number of all numbers greater than 10,000 is equal to......

[22 July 2021 (Shift-II)]

9. If the number of five digit numbers with distinct digits and 2 at the 10^{th} place is $336\,k$, then k is equal to:

[09 June, 2020 (Shift-I)]

(*a*) 7 (*b*) 4
(*c*) 6 (*d*) 8

10. The number of words (with or without meaning) that can be formed from all the letters of the word "LETTER" in which vowels never come together is ________

[06 Sep, 2020 (Shift-II)]

11. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that no two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is_____ **[JEE Adv, 2020]**

12. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is _____ **[JEE Adv, 2020]**

13. Consider a class of 5 girls and 7 boys. The number of different teams consisting of 2 girls and 3 boys that can be formed from this class, if there are two specific boys A and B, who refuse to be the members of the same team, is:

[09 Jan, 2019 (Shift-I)]

(*a*) 500 (*b*) 200 (*c*) 300 (*d*) 350

14. The number of four–digit numbers strictly greater than 4321 that can be formed using the digits 0, 1, 2, 3, 4, 5 (repetition of digits is allowed) is: **[08 April, 2019 (Shift-II)]**

(*a*) 360 (*b*) 288 (*c*) 310 (*d*) 306

FORMATION OF GROUPS AND DISTRIBUTION

15. All the letters of the word "GTWENTY" are written in all possible ways with or without meaning and these words are written as in a dictionary. The serial number of the word "GTWENTY" is _____. **[29 Jan, 2024 (Shift-I)]**

16. There are 4 men and 5 women in Group A, and 5 men and 4 women in Group B. If 4 persons are selected from each group, then the number of ways of selecting 4 men and 4 women is .... **[04 April, 2024 (Shift-II)]**

17. Let $\alpha = \dfrac{(4!)!}{(4!)^{3!}}$ and $\beta = \dfrac{(5!)!}{(5!)^{4!}}$. Then: **[27 Jan, 2024 (Shift-II)]**

(*a*) $\alpha \in N$ and $\beta \notin N$ (*b*) $\alpha \notin N$ and $\beta \in N$
(*c*) $\alpha \in N$ and $\beta \in N$ (*d*) $\alpha \notin N$ and $\beta \notin N$

18. The number of ways of giving 20 distinct oranges to 3 children such that each child gets atleast one orange is_________. **[06 April, 2023 (Shift-I)]**

19. Let n be a non-negative integer. Then the number of divisors of the form "$4n + 1$"of the number $(10)^{10}.(11)^{11}.(13)^{13}$ is equal to_______ **[27 July, 2023 (Shift-II)]**

20. The students $S_1, S_2, \ldots, S_{10}$ are to be divided into 3 groups A, B and C such that each group has at least one student and the group C has at most 3 students. Then the total number of possibilities of forming such groups is

[24 Feb, 2021 (Shift-II)]

CIRCULAR PERMUTATION

21. The number of ways, in which 5 girls and 7 boys can be seated at a round table so that no two girls sit together, is

[08 April, 2023 (Shift-I)]

(*a*) 720 (*b*) $126(5!)^2$ (*c*) $7(360)^2$ (*d*) $7(720)^2$

22. Five person A, B, C, D and E are seated in a circular arrangement. If each of them is given a hat of one of the three colours red, blue and green, then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is **[JEE Adv, 2019]**

TOTAL NUMBER OF COMBINATION AND PERMUTATION

23. The total number of words (with or without meaning) that can be formed out of the letters of the word 'DISTRIBUTION' taken four at a time, is equal to _____. **[31 Jan, 2024 (Shift-I)]**

24. If n is the number of ways five different employees can sit into four indistinguishable offices where any office may have any number of persons including zero, then n is equal to:

[01 Feb, 2024 (Shift-I)]

(*a*) 47 (*b*) 53 (*c*) 51 (*d*) 43

25. There are 5 points P_1, P_2, P_3, P_4, P_5 on the side AB, excluding A and B, of a triangle ABC. Similarly there are 6 points $P_6, P_7, \ldots, P_{11}$ on the side BC and 7 points $P_{12}, P_{13}, \ldots P_{18}$ on the side CA of the triangle. The number of triangles, that can be formed using the points $P_1, P_2, \ldots, P_{18}$ as vertices, is : **[04 April, 2024 (Shift-I)]**

(*a*) 776 (*b*) 751 (*c*) 796 (*d*) 771

26. The number of ways of getting a sum 16 on throwing a dice four times is **[05 April, 2024 (Shift-I)]**

27. The number of triangles whose vertices are at the vertices of regular octagon but none of whose sides is a side of the octagon i

[06 April, 2024 (Shift-I)

(*a*) 24 (*b*) 56
(*c*) 16 (*d*) 48

28. The number of 3-digit numbers, formed using the digits 2, 3, 4, and 7 , when the repetition of digits is not allowed, and which ar not divisible by 3 , is equal to ______ .

[08 April, 2024 (Shift-I

29. The number of ways five alphabets can be chosen from the alphabets of the word MATHEMATICS, where the chosen alphabets are not necessarily distinct, is equal to:

[08 April, 2024 (Shift-II)]

(*a*) 175 (*b*) 181 (*c*) 177 (*d*) 179

30. The number of integers, between 100 and 1000 having the sum of their digits equals to 14, is ______.

[09 April, 2024 (Shift-II)]

31. The lines $L_1, L_2, \ldots, L_{20}$ are distinct. For $n = 1, 2, 3, \ldots, 10$ all the lines L_{2n-1} are parallel to each other and all the lines L_{2n} pass through a given point P. The maximum number of points of intersection of pairs of lines from the set $\{L_1, L_2, \ldots, L_{20}\}$ is equal to: **[01 Feb, 2024 (Shift-II)]**

32. In an examination of Mathematics paper, there are 20 questions of equal marks and the question paper is divided into three sections: A, B and C. A student is required to attempt total 15 questions taking at least 4 questions from each section. If section A has 8 questions, section B has 6 questions and section C has 6 questions, then the total number of ways a student can select 15 questions is ______.

[30 Jan, 2024 (Shift-II)]

33. The number of ways in which 21 identical apples can be distributed among three children such that each child gets at least 2 apples, is

[31 Jan, 2024 (Shift-II)]

(*a*) 406 (*b*) 130 (*c*) 142 (*d*) 136

34. If the number of words, with or without meaning, which can be made using all the letters of the word MATHEMATICS in which C and S do not come together, is $(6!)k$, then k is equal to **[08 April, 2023 (Shift-II)]**

(*a*) 1890 (*b*) 945 (*c*) 2835 (*d*) 5670

35. Some couples participated in a mixed doubles badminton tournament. If the number of matches played, so that no couple played in a match, is 840, then the total numbers of persons, who participated in the tournament, is______.

[10 April, 2023 (Shift-I)]

36. A class contains b boys and g girls. If the number of ways of selecting 3 boys and 2 girls from the class is 168, then $b + 3g$ is equal to______. **[28 July, 2022 (Shift-II)]**

37. Numbers are to formed between 1000 and 3000, which are divisible by 4, using the digits 1, 2, 3, 4, 5 and 6 without repetition of digits. Then the total number of such numbers is____. **[26 July, 2022 (Shift-II)]**

38. Consider a rectangle $ABCD$ having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the number of triangles having these points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta - \alpha)$ is equal to:

[16 March, 2021 (Shift-II)]

(*a*) 795 (*b*) 1173 (*c*) 1890 (*d*) 717

39. All the arrangements, with or without meaning, of the word FARMER are written excluding any word that has two R appearing together. The arrangements are listed serially in the alphabetic order as in the English dictionary. Then the serial number of the word FARMER in this list is ______.

[01 Sep, 2021 (Shift-II)]

40. Let $S_1 = \{(i, j, k) : i, j, k \in \{1, 2, \ldots, 10\}\}$

$S_2 = \{(i, j) : 1 \le i < j + 2 \le 10, i, j \in \{1, 2, \ldots, 10\}\}$,

$S_3 = \{(i, j, k, l) : 1 \le i < j < k < l, i, j, k, l \in \{1, 2, \ldots, 10\}\}$.

and $S_4 = \{(i, j, k, l)$: i, j, k and l are distinct elements in $\{1, 2, \ldots, 10\}\}$.

If the total number of elements in the set S_r is n_r, $r = 1, 2, 3, 4$, then which of the following statements is (are) TRUE?

[JEE Adv, 2021]

(*a*) $n_1 = 1000$ (*b*) $n_2 = 44$

(*c*) $n_3 = 220$ (*d*) $n_4/12 = 420$

41. There are 3 sections in a question paper and each section contains 5 questions. A candidate has to answer a total of 5 questions, choosing at least one question from each section. Then the number of ways, in which the candidate can choose the questions, is: **[05 Sep, 2020 (Shift-II)]**

(*a*) 2255 (*b*) 3000 (*c*) 2250 (*d*) 1500

42. A test consists of 6 multiple choice questions, each having 4 alternative answers of which only one is correct. The number of ways, in which a candidate answers all six questions such that exactly four of the answers are correct, is ______.

[04 Sep, 2020 (Shift-II)]

43. The number of functions f from $\{1, 2, 3, \ldots, 20\}$ only $\{1, 2, 3, \ldots, 20\}$ such that $f(k)$ is a multiple of 3, whenever k is a multiple of 4, is: **[11 Jan, 2019 (Shift-II)]**

(*a*) $6^5 \times (15)!$ (*b*) $5! \times 6!$

(*c*) $(15)! \times 6!$ (*d*) $5^6 \times 15$

DISTRIBUTION OF ALIKE AND DIFFERENT OBJECTS

44. Number of ways of arranging 8 identical books into 4 identical shelves where any number of shelves may remain empty is equal to **[29 Jan, 2024 (Shift-II)]**

(*a*) 18 (*b*) 16 (*c*) 12 (*d*) 15

45. The number of triplets (x, y, z) where x, y, z are distinct non negative integers satisfying $x + y + z = 15$, is

[11 April, 2023 (Shift-I)]

(*a*) 80 (*b*) 114 (*c*) 92 (*d*) 136

46. The number of ways of choosing 10 objects out of 31 objects of which 10 are identical and the remaining 21 are distinct, is: **[12 April, 2019 (Shift-I)]**

(*a*) 2^{20} (*b*) $2^{20} + 1$ (*c*) 2^{21} (*d*) $2^{20} - 1$

DERANGEMENT

47. In an examination, 5 students have been allotted their seats as per their roll numbers. The number of ways, in which none of the students sits on the allotted seat, is ________.

[11 April, 2023 (Shift-I)]

Credit:--- Divansh.....

PW Challengers

1. For any finite non empty set X of integers, let $\max(X)$ denote the largest element of X and $|X|$ denote the number of elements in X. If N is the number of ordered pairs, (A, B) of finite non-empty sets of positive integers, such that

   $\max(A) \times |B| = 12$; and

   $|A| \times \max(B) = 11$

   and N can be written as $100a + b$ where a, b are positive integers less than 100, find $a + b$.

2. Find the number of triples (a, b, c) of positive integers such that,
   - (*i*) ab is a prime;
   - (*ii*) bc is a product of two primes;
   - (*iii*) abc is not divisible by square of any prime and
   - (*iv*) $abc \leq 30$

3. Consider the set $S = \{(a, b, c, d, e); 0 < a < b < c < d < e < 100\}$ where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of S, find the largest integer less than or equal to D.

4. In the coordinate plane consider the set S of all points with integer coordinates. For a positive integer k, two distinct points $A, B \in S$ will be called *k-friends* if there is a point $C \in S$ such that the area of the triangle ABC is equal to k. A set $T \in S$ will be called a *k-clique* if every two points in T are *k-friends*. Find the least positive integer k for which there exists a *k-clique* with more than 200 elements.

5. Let n and k be fixed positive integers of the same parity, $k \geq n$. We are given $2n$ lamps numbered 1 through $2n$; each of them can be *on* or *off*. At the beginning all lamps are *off*. We consider sequences of k steps. At each step one of the lamps is switched (from *off* to *on* or from *on* to *off*).

   Let N be the number of k-step sequences ending in the state: lamps $1, \ldots, n$ on, lamps $n + 1$ ..... $2n$ *off*.

   Let M be the number of k-step sequences leading to the same state and not touching lamps $n + 1$ ..... $2n$ at all.

   Find the ratio N/M, for $(k, n) = (2025, 2015)$.

6. Find all positive integers n, for which the numbers in the set $S = (1, 2 \ldots., n)$ can be colored red and blue, with the following condition being satisfied: the set $S \times S \times S$ contains exactly 2007 ordered triples (x, y, z) such that (*i*) x, y, z are of the same color and (*ii*) $x + y + z =$ is divisible by n, find sum of solutions of (*i*) and (*ii*).

7. A diagonal of a regular 2006-gon is called *odd* if its endpoints divide the boundary into two parts, each composed of an odd number of sides. Sides are also regarded as odd diagonals.

   Suppose the 2006-gon has been dissected into triangles by 2003 nonintersecting diagonals. Find the maximum possible number of isosceles triangles with two odd sides.

8. If all the integral pairs (x, y) satisfying the equation $1 + 2^x + 2^{2x+1} = y^2$ are (α_i, β_i) then find $\Sigma(\alpha_i + \beta_i)$ (where $\alpha_i < \alpha_{i+1}$ and $\beta_i < \beta_{i+1}$)

9. If (k, n) is pair of positive integers for which $7^k - 3^n$ divides $k^4 + n^2$, then find sum of all the possible values of ${}^nC_k$

10. We are given an infinite deck of cards, each with a real number on it. For every real number r, there is exactly one card in the deck that has a written on it. Now two players draw disjoint sets A and B of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:
    1. The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.
    2. If we write the elements of both sets in increasing order as $A = (a_1, a_2 \ldots a_{100})$ and $B = (b_1, b_2 \ldots b_{100})$, and $a_i > b_i$ for all i, then A beats B.
    3. If three players draw three disjoint sets A, B, C from the deck. A beats B and B beats C, then A also beats C.

    How many ways are there to define such a rule? (Here, we consider two rules as different if there exist two sets A and B such that A beats B according to one rule, but B beats A according to the other.)

Credit:--- Divansh.....

ANSWER KEY

CONCEPT APPLICATION

1. (i) 9, (ii) 20, (iii) 72, (iv) 326592 2. (b) 3. (a) 4. (a) 5. (d) 6. (b) 7. (i) (a), (ii) (a), (iii) (a)
8. (b) 9. (c) 10. (c) 11. (b) 12. (a) 13. (a) 14. (c)
15. (i) 50, (ii) 205, (iii) 60, (iv) 378, (v) 270 16. (d) 17. (c) 18. [2520] 19. [25920]
20. (i) $5 \times 6 \times 2^3$, (ii) $5 \times 6 \times 2^3 - 1$, (iii) $4 \times 5 \times (2^3 - 1)$ 21. [119] 22. [21] 23. (c)
24. (i) 1440, (ii) 1438, (iii) 480, (iv) 960, (v) $5 \cdot 7 \cdot 11 \cdot 9 \cdot 7$, (vi) $\frac{2^3-1}{2-1} \cdot \frac{3^4-1}{3-1} \cdot \frac{5^6-1}{5-1} \cdot \frac{7^5-1}{7-1} \cdot \frac{11^4-1}{11-1}$, (vii) 720, (viii) 16, (ix) 540, (x) 480, (xi) 50
25. (a,b,c,d) 26. (b) 27. (b) 28. (b) 29. $\frac{30!}{10!10!10!}\frac{1}{3!}$ 30. $\frac{30!}{10!10!10!3!}3!$
31. (a) 32. [10] 33. [3150] 34. (b,d) 35. $({}^{23}C_2)$ 36. (b) 37. (c) 38. (a) 39. (b)
40. [20] 41. (d)

SCHOOL LEVEL PROBLEMS

1. (d) 2. (a) 3. (c) 4. (c) 5. (c) 6. (c) 7. (d) 8. (d) 9. (c) 22. (c)
23. (b) 24. (a) 25. (a)

PRARAMBH (TOPICWISE)

1. (b) 2. (d) 3. (a) 4. (c) 5. (c) 6. (d) 7. (d) 8. (a) 9. (a) 10. (d)
11. (b) 12. (d) 13. (b) 14. (c) 15. (c) 16. (a) 17. (a) 18. (a) 19. (b) 20. (a)
21. (b) 22. (c) 23. (d) 24. (c) 25. (a) 26. (d) 27. (a) 28. (a) 29. (c) 30. (c)
31. (d) 32. (a) 33. (c) 34. (b) 35. (b) 36. (c) 37. (b) 38. (b) 39. (a) 40. (c)
41. (b) 42. (a) 43. (a) 44. (a) 45. (a) 46. (d) 47. (a) 48. (b) 49. (b) 50. (c)

PRABAL (JEE MAIN LEVEL)

1. (c) 2. (b) 3. (b) 4. (b) 5. (c) 6. (b) 7. (b) 8. (a) 9. (b) 10. (a)
11. (c) 12. (b) 13. (b) 14. (b) 15. (a) 16. (b) 17. (a) 18. (d) 19. (a) 20. (b)
21. (b) 22. (a) 23. (b) 24. (b) 25. (c) 26. (a) 27. (a) 28. (d) 29. (b) 30. (c)
31. (d) 32. (c) 33. (b) 34. (c) 35. [6] 36. [420] 37. [15] 38. [1750] 39. [12] 40. [1]
41. [159] 42. [2220] 43. [5292] 44. [60] 45. [6] 46. [5] 47. [315] 48. [21599]

PARIKSHIT (JEE ADVANCED LEVEL)

1. (a) 2. (b) 3. (a) 4. (a) 5. (b) 6. (a) 7. (b) 8. (b) 9. (b) 10. (b)
11. (a) 12. (b,d) 13. (a,b,c,d) 14. (a,c,d) 15. (b,c,d) 16. (a,d) 17. (b,c,d) 18. (a,b,d) 19. (a,d) 20. (b,c,d)
21. (a,c) 22. (b,c) 23. (b,c,d) 24. (a,b,d) 25. (b,c,d) 26. (b,d) 27. (a) 28. (b) 29. (d) 30. (b)
31. (a) 32. (a) 33. (a) 34. (a) 35. (b) 36. (b) 37. (b) 38. (a) 39. (a) 40. (c)
41. (b) 42. (d) 43. (b) 44. [7] 45. [4] 46. [20] 47. [140] 48. [250] 49. [2220] 50. [462]
51. [4]

PYQ's (PAST YEAR QUESTIONS)

1. (c) 2. (c) 3. (a) 4. [432] 5. (d) 6. [569] 7. (b) 8. [96] 9. (d) 10. [120]
11. [495] 12. [1080] 13. (c) 14. (c) 15. [553] 16. [5626] 17. (c) 18. [3483638676] 19. [924]
20. [31650] 21. (b) 22. [30] 23. [3734] 24. (c) 25. (b) 26. [125] 27. (c) 28. [36] 29. (d)
30. [70] 31. [101] 32. [11376] 33. (d) 34. (d) 35. [16] 36. [17] 37. [30] 38. (d) 39. [77]
40. (a,b,d) 41. (c) 42. [135] 43. (c) 44. (d) 45. (b) 46. (a) 47. [44]

PW CHALLENGERS

1. [43] 2. [17] 3. [66] 4. [180180] 5. 1024 6. [153] 7. [1003] 8. [8] 9. (6) 10. [100]

Credit:--- Divansh.....